Some examples and a note on cubics, 316-IX

February 19, 2003

1 Problem 4.8.3

Find a particular solution to the DE

$$2z''(t) + z(t) = 9e^{2x} \ .$$

Solution:
The characteristic equation is $2r^2 + 1 = 0$ with roots $r = \pm\frac{i}{2}$. Since -2 is not a root, try

$$\begin{aligned} z_p(x) &= Ae^{2x} \\ z_p'(t) &= 2Ae^{2x} \\ z_p''(t) &= 4Ae^{2x} \end{aligned}$$

so that

$$z'' - z = (8A + A)e^{2x} = 9e^{2x}$$

i.e.

$$A = 1 \rightarrow z_p(x) = e^{2x} \ .$$

2 Problem 4.8.34

Determine the form of a particular solution to the DE

$$y'' - y = e^{2x} - xe^{2x} + x^2e^{2x} \ .$$

Solution:
The characteristic equation is $r^2 - 1 = 0$ with roots $r = 1, -1$. Since 2 is not a root, try

$$\begin{aligned} y_p(x) &= e^{2x}(Ax^2 + Bx + C) \\ y_p'(x) &= e^{2x}\left[2Ax^2 + 2(A+B)x + (B+2C)\right] \\ y_p''(x) &= e^{2x}\left[4Ax^2 + 4(2A+B)x + 2(A+2B+2C)\right] \end{aligned}$$

so that

$$x'' - x = e^{2x} - xe^{2x} + x^2e^{2x} = e^{2x}(x^2 - x + 1)$$

i.e.

$$4Ax^2 + 4(2A+B)x + 2(A+2B+2C) - (Ax^2 + Bx + C) = x^2 - x + 1$$

i.e.

$$\begin{aligned} 3A &= 1 \\ 8A + 3B &= -1 \\ 2A + 4B + 3C &= 1 \end{aligned}$$

and solving:

$$A = 1/3 \ , \ B = -11/9 \ , \ C = 47/27$$

and

$$y_p(t) = e^{2x}\left(\frac{1}{3}x^2 - \frac{11}{9}x + \frac{47}{27}\right) \ .$$

```
> restart;with(DEtools):with(linalg):with(plots):

Warning, the name adjoint has been redefined

Warning, the protected names norm and trace have been redefined and
unprotected

Warning, the name changecoords has been redefined
```

An example of using Maple to find solutions to IVP for a given ODE (Text, problem 4.8.37).

Consider the ODE (linear, homogeneous, constant coefficients):

```
> de1 := D(D(y))(x) - 4*D(y)(x) + 5*y(x) =
> exp(5*x)+x*sin(3*x)-cos(3*x);
```

$$de1 := (\mathrm{D}^{(2)})(y)(x) - 4\,\mathrm{D}(y)(x) + 5\,\mathrm{y}(x) = e^{(5\,x)} + x\sin(3\,x) - \cos(3\,x)$$

The initial conditions are

```
> init_con := y(0) = 0, D(y)(0) = 0;
```

$$init\_con := \mathrm{y}(0) = 0,\ \mathrm{D}(y)(0) = 0$$

We now compute the solution using DSOLVE (set C1=C2=0 for particular solution):

```
> gsolution := dsolve({de1},y(x));
```

$$gsolution := \{\mathrm{y}(x) = e^{(2\,x)}\sin(x)\,\_C2 + e^{(2\,x)}\cos(x)\,\_C1 + \frac{1}{400}(16+30\,x)\cos(3\,x) + \frac{1}{10}e^{(5\,x)} + \frac{13}{400}\sin(3\,x) - \frac{1}{40}x\sin(3\,x)\}$$

Now some MAPLE esoterica: convert the FUNCTION y(x) to an EXPRESSION that can be evaluated and plotted;

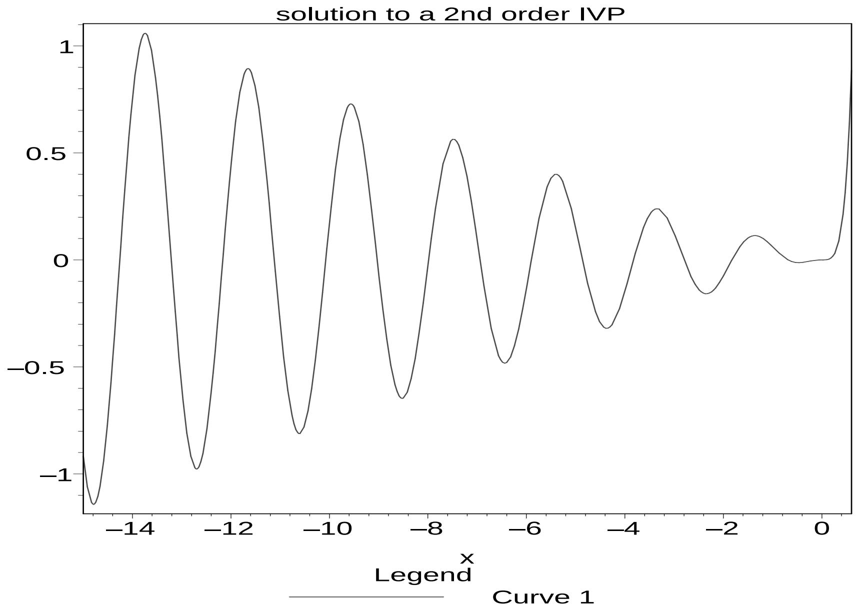
this is done with the command subs:

```
> IVP := {de1,init_con};soln := dsolve(IVP, y(x));expr :=
> subs(soln,y(x)):
```

$$IVP := \{(\mathrm{D}^{(2)})(y)(x) - 4\,\mathrm{D}(y)(x) + 5\,\mathrm{y}(x) = e^{(5\,x)} + x\sin(3\,x) - \cos(3\,x),\ \mathrm{D}(y)(0) = 0,\ \mathrm{y}(0) = 0\}$$

$$soln := \mathrm{y}(x) = -\frac{157}{400}e^{(2\,x)}\sin(x) - \frac{7}{50}e^{(2\,x)}\cos(x) + \frac{1}{400}(16+30\,x)\cos(3\,x) + \frac{1}{10}e^{(5\,x)} + \frac{13}{400}\sin(3\,x) - \frac{1}{40}x\sin(3\,x)$$

```
> plot(expr,x=-15..0.6, axes=BOXED,title="solution to a 2nd order
> IVP");
```



3 Problem 4.8.45

Find a particular solution to the higher order DE

$$y''' - y'' + y = sinx \ .$$

Solution:
The characteristic equation is $r^3 - r^2 + 1 = 0$. It is easy to check that i is not a root.

4 Problem 4.8.48

Find a particular solution to the higher order DE

$$y^{iv} - y'' - 8y = sinx \ .$$

Solution:
The characteristic equation is $r^4 - r^2 - 8 = 0$. It is easy to check that i is not a root.

5 The solution of the cubic $r^3 + ar^2 + br + c = 0$

Suppose you wanted to solve

$$y''' + ay'' + by' + cy = 0$$

then substituting $y = e^{rt}$ results in the cubic

$$r^3 + ar^2 + br + c = 0$$

Let

$$r = z - A$$

and plug in to get

$$\begin{aligned} (z^3 - 3Az^2 + 3A^2z - A^3) + a(z^2 - 2Az + A^2) + b(z - A) + c &= 0 \\ z^3 + (-3A + a)z^2 + (3A^2 - 2Aa + b)z + (-A^3 + aA^2 - bA + c) &= 0 \end{aligned}$$

Choose

$$A = a/3$$

to eliminate the quadratic term. Then, equation becomes:

$$z^3 + Bz + C = 0 \ , \ B = b - \frac{a^2}{3} \ , \ C = \frac{2a^3}{27} - \frac{ab}{3} + c$$

How can we solve this equation for z? Let

$$z = x + y$$

then

$$z^3 + Bz + C = 0 \Rightarrow (x + y)^3 + B(x + y) + C = 0$$

or

$$x^3 + 3x^2y + 3xy^3 + y^3 + B(x + y) + C = 0 \Rightarrow x^3 + y^3 + (3xy + B)(x + y) + C = 0$$

Now let

$$3xy + B = 0$$

(since we replaced one unknown (z) by two (x and y) we have some freedom in the choice!) Then x and y satisfy:

$$3xy + B = 0 \Rightarrow x^3y^3 = -\frac{B^3}{27}$$

while the cubic becomes

$$x^3 + y^3 + C = 0$$

Now let

$$U = x^3 \text{ and } V = y^3$$

Then

$$UV = -\frac{B^3}{27} \ , \ U + V = -C$$

But two numbers whose sum and product are given can be found as the roots of a quadratic equation since

$$(x-a)(x-b) = x^2 - (a+b)x + ab = 0$$

has roots a, b. Similarly then U, V are roots of the quadratic:

$$t^2 + Ct - \frac{B^3}{27} = 0$$

That is:

$$t = -\frac{C}{2} \pm \sqrt{\left(\frac{C}{2}\right)^2 + \frac{B^3}{27}}$$

Then

$$U = t_+ \ , \ V = t_-$$

and

$$x = U^{\frac{1}{3}} \ , \ y = V^{\frac{1}{3}} = -\frac{B}{3x}$$

and

$$z = x + y = x - \frac{B}{3x}$$

5.1 Example: $r^3 - 3r - 2 = 0$

Now $B = 3$, $C = -2$ so

$$t = 1 \pm \sqrt{1-1} = 1$$

and

$$x_1 = 1 \ , \ x_2 = e^{i\frac{2\pi}{3}} \ , \ x_3 = e^{-i\frac{2\pi}{3}}$$

$$r_1 = 1 + 1 = 2 \ , r_2 = e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}} \ , \ r_3 = e^{-i\frac{2\pi}{3}} + e^{i\frac{2\pi}{3}} = r_2$$

That is both r_2 and r_3 are real! It is remarkable that in order to find them we had to resort to complex numbers, but the end result is nevertheless real! We get

$$r_2 = r_3 = 2\cos\frac{2\pi}{3} = -1$$

and

$$r^3 - 3r - 2 = (r-2)(r+1)^2 = 0$$

5.2 Example: y"' - y" + y = 0

$$r^3 - r^2 + 1 = 0 \ , \ a = -1 \ ; \ b = 0 \ ; \ c = 1$$

Then

$$r = z - \frac{a}{3} = z + \frac{1}{3}$$

and z satisfies

$$z^3 + Bz + C = 0 \ , \ B = b - \frac{a^2}{3} = -\frac{1}{3} \ , \ C = \frac{2a^3}{27} - \frac{ab}{3} + c = -\frac{2}{27} + 1 = \frac{25}{27}$$

$$z^3 - \frac{1}{3}z + \frac{25}{27} = 0$$

The associated quadratic is

$$t^2 + Ct - \frac{B^3}{27} = 0 \Rightarrow t^2 + \frac{25}{27}t + \frac{1}{27^2} = 0$$

with roots

$$t = \left(\frac{1 \pm \sqrt{\frac{23}{3}}}{6}\right)$$

Then, x is a cube root of t_+ or, if we define

$$d = \left[\frac{1 + \sqrt{\frac{23}{3}}}{6}\right]^{\frac{1}{3}}$$

then

$$x_1 = d \ , \ x_2 = de^{i\frac{2\pi}{3}} \ , \ x_3 = de^{-i\frac{2\pi}{3}}$$

and

$$z_1 = d + \frac{1}{9d} \ , \ z_2 = de^{i\frac{2\pi}{3}} + \frac{1}{9d}e^{-i\frac{2\pi}{3}} \ , \ z_3 = de^{-i\frac{2\pi}{3}} + \frac{1}{9d}e^{i\frac{2\pi}{3}}$$

and finally

$$r_i = z_i + \frac{1}{3} \ .$$

The homogeneous solution to the cubic is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + C_3 e^{r_3 t}$$

where it is clear from the above that two of the roots are complex conjugates while the other is real. That is,

$$r_2 = u + iv \ , \ r_3 = u - iv$$

so the homo. solution has the form

$$y(t) = C_1 e^{r_1 t} + e^{ut} \left(C_2 \cos vt + C_3 \sin vt \right) .$$