Solutions, 316-XI (S'04)

April 22, 2004

1 Problem 5.2.2

Find general solution and plot the direction field:

$$x_1' = 2x_1 + 3x_2 \ , \ x_2' = 2x_1 + x_2 \ .$$

Solution: Vector form:

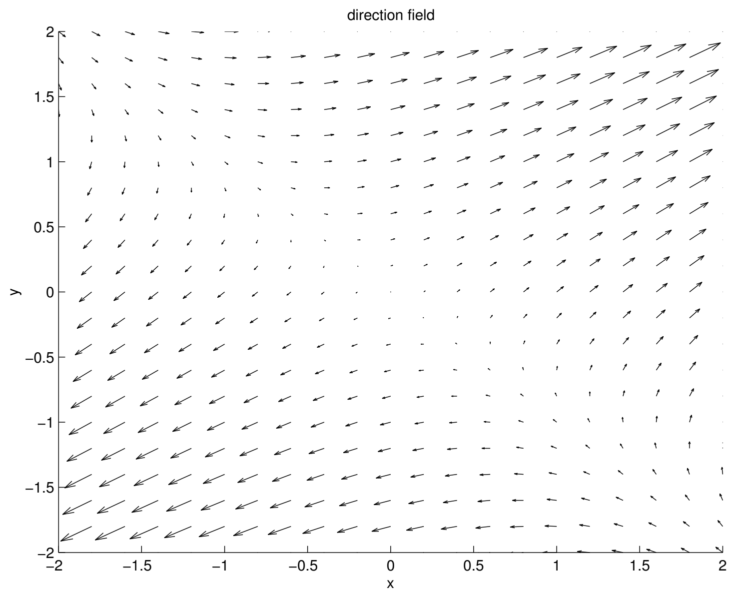
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{rt}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 2-r & 3 \\ 2 & 1-r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad r^2 - 3r - 4 = 0 \rightarrow r = \frac{3 \pm 5}{2}$$

$$r_1 = 4 \ , -2v_1 + 3v_2 = 0 \rightarrow \mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad , \quad r_2 = -1 \ , 3v_1 + 3v_2 = 0 \rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathbf{x}(t) = C_1 e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad + \quad C_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

```
function draw()
axis([-2 2 -2 2]); X1 = linspace(-2,2,21); Y1 = linspace(-2,2,21);
hold on
pp(@f,@g,X1,Y1,'x','y','direction field','r',0);
function xp = f(x,y)
xp = 2*x +3*y ;
function yp = g(x,y)
yp = 2*x + y;
```



2 Problem 5.2.9

Find general solution and solution to IVP and plot the direction field

$$x_1' = 2x_1 - 5x_2 \ , \ x_2' = 4x_1 - 2x_2 \ ; \ x_1(0) = 2 \ , x_2(0) = 3 \ .$$

Solution:
Vector form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{rt}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 2-r & -5 \\ 4 & -2-r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad r^2 + 16 = 0 \rightarrow r = \pm 4i$$

$$r_\pm = \pm 4i \ , (2 \mp 4i)v_1 - 5v_2 = 0 \quad \rightarrow \quad \mathbf{v}_\pm = \begin{pmatrix} 5 \\ 2 \mp 4i \end{pmatrix}$$

$$\mathbf{v}_\pm = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \mp i \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad \rightarrow \quad \mathbf{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \ , \ \mathbf{b} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

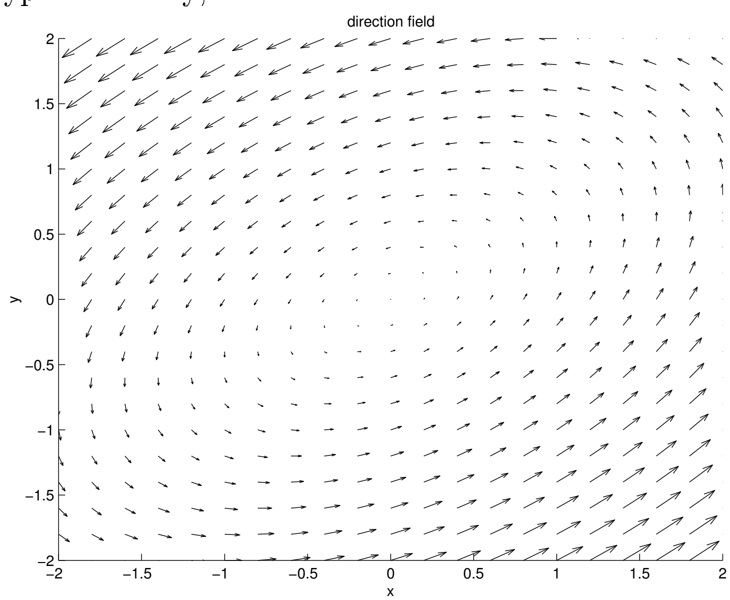
$$\mathbf{x}(t) = C_1 \begin{pmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{pmatrix} \quad + \quad C_2 \begin{pmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{pmatrix}$$

At $t = 0$:

$$\mathbf{x}(0) = C_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 5 & 0 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{-20} \begin{pmatrix} -4 & 0 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8/20 \\ -11/20 \end{pmatrix} \quad \rightarrow \quad \mathbf{x} = \begin{pmatrix} 2\cos 4t - \frac{11}{4}\sin 4t \\ 3\cos 4t + \frac{1}{2}\sin 4t \end{pmatrix}$$

```
function draw()
axis([-2 2 -2 2]); X1 = linspace(-2,2,21); Y1 = linspace(-2,2,21);
hold on
pp(@f,@g,X1,Y1,'x','y','direction field','r',0);
function xp = f(x,y)
xp = 2*x -5*y ;
function yp = g(x,y)
yp = 4*x -2*y;
```



3 Problem 5.2.12

Find general solution and plot the direction field:

$$x_1' = x_1 - 5x_2 \ , \ x_2' = x_1 + 3x_2 \ .$$

Solution: Vector form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \to \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{rt}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

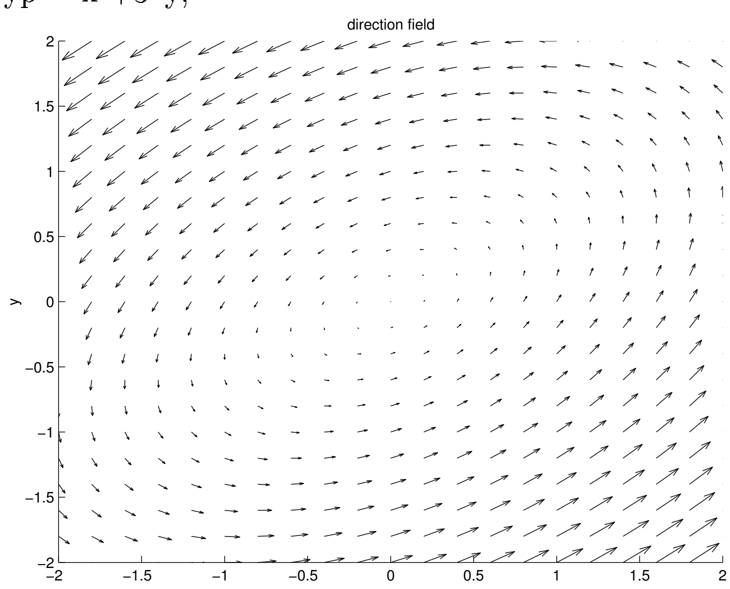
$$\begin{pmatrix} 1-r & -5 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad r^2 - 4r + 8 = 0 \to r = \frac{4 \pm 4i}{2}$$

$$r_\pm = 2 \pm 2i \; , (-1 \mp 2i)v_1 - 5v_2 = 0 \quad \to \quad \mathbf{v}_\pm = \begin{pmatrix} 5 \\ -1 \mp 2i \end{pmatrix}$$

$$\mathbf{v}_\pm = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \mp i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \to \quad \mathbf{a} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \; , \; \mathbf{b} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\mathbf{x}(t) = C_1 e^{2t}\begin{pmatrix} 5\cos 2t \\ -1\cos 2t + 2\sin 2t \end{pmatrix} \quad + \quad C_2 e^{2t}\begin{pmatrix} 5\sin 2t \\ -1\sin 2t - 2\cos 2t \end{pmatrix}$$

```
function draw()
axis([-2 2 -2 2]); X1 = linspace(-2,2,21); Y1 = linspace(-2,2,21);
hold on
pp(@f,@g,X1,Y1,'x','y','direction field','r',0);
function xp = f(x,y)
xp = x -5*y ;
function yp = g(x,y)
yp = x +3*y;
```



4 Problem 5.2.41

The coefficient matrix of the 4×4 system

$$\begin{array}{rcrcrcrcr} x_1' & = & 4x_1 & + & x_2 & + & x_3 & + & 7x_4 \\ x_2' & = & x_1 & + & 4x_2 & + & 10x_3 & + & x_4 \\ x_3' & = & x_1 & + & 10x_2 & + & 4x_3 & + & x_4 \\ x_4' & = & 7x_1 & + & x_2 & + & x_3 & + & 4x_4 \end{array}$$

has eigenvalues $r_1 = -3$, $r_2 = -6$, $r_3 = 10$ and $r_4 = 15$. Find the particular solution of this system that satisfies the initial conditions

$$x_1(0) = 3 \ , \ x_2(0) = x_3(0) = 1 \ , \ x_4(0) = 3 \ .$$

Solution:
The matlab script **syss.m** provided, with matrix:
A = [4 1 1 7; 1 4 10 1; 1 10 4 1; 7 1 1 4];
and initial vector:
x0 = [3;1;1;3];
produces the output:

$$\begin{array}{rcrrrr} D & = & -6.0000 & 0 & 0 & 0 \\ & & 0 & -3.0000 & 0 & 0 \\ & & 0 & 0 & 10.0000 & 0 \\ & & 0 & 0 & 0 & 15.0000 \end{array}$$

$$\begin{array}{rcrrrr} V & = & 0.0000 & -1.0000 & -2.0000 & -1.0000 \\ & & -1.0000 & -0.0000 & 1.0000 & -2.0000 \\ & & 1.0000 & 0.0000 & 1.0000 & -2.0000 \\ & & -0.0000 & 1.0000 & -2.0000 & -1.0000 \end{array}$$

$$\begin{array}{rcr} C & = & 0.0000 \\ & & -0.0000 \\ & & -1.0000 \\ & & -1.0000 \end{array}$$

so that the particular solution is

$$\mathbf{x}(t) = -e^{10t} \begin{pmatrix} -2 \\ 1 \\ 1 \\ -2 \end{pmatrix} - e^{15t} \begin{pmatrix} -1 \\ -2 \\ -2 \\ -1 \end{pmatrix}$$

5 Problem 5.2.45

Find the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with matrix

$$\mathbf{A} = \begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix}$$

Solution:
The matlab script **syss.m**, with matrix:
A = [9 -7 -5 0;-12 7 11 9;24 -17 -19 -9;-18 13 17 9];
gives:

$$D = \begin{bmatrix} -3.0000 & 0 & 0 & 0 \\ 0 & 6.0000 & 0 & 0 \\ 0 & 0 & 0.0000 & 0 \\ 0 & 0 & 0 & 3.0000 \end{bmatrix}$$

$$V = \begin{bmatrix} 1.0000 & -1.0000 & 1.0000 & 2.0000 \\ 1.0000 & 1.0000 & 2.0000 & 1.0000 \\ 1.0000 & -2.0000 & -1.0000 & 1.0000 \\ -1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

so that the general solution is:

$$\mathbf{x}(t) = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{3t} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} .$$

6 Problem 5.4.3

Find the general solution of the system:

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \mathbf{x} \ .$$

Solution:
The eigenvalues:

$$\det \begin{bmatrix} 1-r & -2 \\ 2 & 5-r \end{bmatrix} = 0 \rightarrow r^2 - 6r + 9 = (r-3)^2 = 0 \rightarrow r = 3 \ \ \text{(double)}$$

The eigenvector:

$$(1-r)v_1 - 2v_2 = 0 \rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The generalized eigenvector:

$$(1-r)u_1 - 2u_2 = v_1 \ , \ 2u_1 + (5-r)u_2 = v_2 \rightarrow \ \text{let } u_2 = 0 \text{ then } u_1 = -1/2$$

i.e.

$$\mathbf{u} = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$$

so that the general solution is

$$\mathbf{x}(t) = e^{3t} \left[C_1 \mathbf{v} + C_2 \left(t\mathbf{v} + \mathbf{u} \right) \right] \ .$$

7 Problem 5.4.4

Find the general solution of the system:

$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \mathbf{x} \ .$$

Solution:
The eigenvalues:

$$\det \begin{bmatrix} 3-r & -1 \\ 1 & 5-r \end{bmatrix} = 0 \to r^2 - 8r + 16 = (r-4)^2 = 0 \to r = 4 \quad \text{(double)}$$

The eigenvector:

$$(3-r)v_1 - v_2 = 0 \to \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The generalized eigenvector:

$$(3-r)u_1 - u_2 = v_1 \ , \ \ u_1 + (5-r)u_2 = v_2 \to \ \text{let } u_2 = 0 \text{ then } u_1 = -1$$

i.e.

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so that the general solution is

$$\mathbf{x}(t) = e^{4t}\left[C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{u}\right)\right] \ .$$