Use of Tools from *Interactive Differential Equations*
with the texts
***Fundamentals of Differential Equations*, 5th edition**
and
***Fundamentals of Differential Equations and Boundary Value Problems*, 3rd edition**
by Nagle, Saff, and Snider

Chapter One

Chapter 1, Section 1.3, Exercise 10, page 23. Use the **Slope Fields Tool**. Pick an equation like $\dot{x} = x - t$. Set the tool drawing mode to "Vectors" and v*(t)* to .50. Click the mouse at any initial position in the left window and, on the head of each arrow that occurs, click the mouse again. See if you can determine the configuration of the resulting solution curve. Set the drawing mode to "Solutions" and click the mouse on the tail of the first arrow. Does the resulting solution curve resemble what you imagined? Clear the field and set Δt to be 0.10 and repeat. Note the differences and explain why.

Chapter 1, Section 1.5, Example 2, page 34. Use the **Solutions Tool**. Pick the equation $\dot{x} = x$ and the Euler method on the bottom left. For Δt successively equal to 0.5, 0.1, 0.05, and 0.01, click on an initial point in the screen and compare the actual solution with the numerical solution generated by Euler's method. What differences do you notice? Try it again for another equation.

Chapter 1, Section 1.4, Exercises 1-10, page 31. Use the **Logistic Phase Line Tool**. Click at an initial point on the direction field. Observe the trajectory and determine to what values it converges as *t* grows in value. Identify the equilibrium points and classify them as sinks, sources, or nodes.

Chapter 1, Section 1.3, The Method of Isoclines, page 20. Use the **Isoclines Tool**. First click on the clear button. Pick any equation from the menu. Slide to various values of *c* on the *c* scale. Notice that on the lines that are drawn, all the arrows have the same slope. Click on the graph to plot a trajectory and notice how the trajectory fits into the direction field created by the isoclines.

Chapter 1, Section 1.2, Example 9 (page 13) and Exercise 29 (page 15). Use the **Uniqueness Tool**. Select the option $\dot{x} = x^{2/3}$ from the menu. Click anywhere on the left graph. Do any of the solutions intersect? How about if you click anywhere on the *t* - axis in addition? What does that say about when the solutions to initial value problems involving $\dot{x} = x^{2/3}$ are unique?

Chapter Two

Chapter 2, Section 2.2, Exercise 34, Newton's Law of Cooling, page 53. Use **Newton's Law of Cooling: Cooling Rate Tool.** First click the mouse on a point in the upper-left graphical window. Notice the general shapes of all three curves on the screen. Experiment with various values for the constant of proportionality k and the temperature of the medium A. What happens to T and to $\dot{T} = \frac{dT}{dt}$ as t becomes very large? What is the slope of the straight line in the graph of $\dot{T}$ versus $A - T$?

Chapter 2, Section 2.2, Exercise 38, Free Fall, page 53. Use **Falling Bodies Tool**. Select various values on the sliders for the mass m and the constant b . Observe how the graphs of $v(t)$ and $y(t)$ change as these constants change. From the differential equation given in Exercise 38 and the graphs given in this tool, what is a formula for the terminal velocity of the falling object?

Chapter 2, Section 2.4, Exercise 32, Orthogonal Trajectories, page 71. Use the **Orthogonal Trajectories Tool**. From the menu of systems select the third, which has for its trajectories families of circles, and notice the orthogonal trajectories. Also select the second option, which has for its trajectories the hyperbolas $xy = k$, and notice the orthogonal trajectories. Experiment with other orthogonal trajectories as well.

Chapter Three

Chapter 3, Section 3.3, Exercise 1, page 113. **Use Newton's Law of Cooling: Curve Fitting Tool**. Click on the "Show Coffee Data" button. By fitting a curve to the data shown determine the correct values for the room temperature A and the constant of proportionality k . At what time is the coffee at 120 degrees?

Chapter 3, Section 3.5, Euler's Method, page 126. Use the **Time Steps Tool**. Click on the $\Delta t = 1$ button. The computer takes a very large step from $t = 0$ to $t = 1$. What is the slope of this step? Then at $t = 1$, the computer computes the local slope and takes another very large step to $t = 2$. What is the slope of this step? What is the prediction of the computer at $t = 2$? How close is it to the actual value of the solution? Examine what happens for smaller values of Δt .

Chapter 3, Section 3.6, page 135. Use the **Numerical Methods Tool**. There are five different numerical methods contrasted here. Click successively on the buttons from left to right corresponding to each one and observe variations of accuracy to the actual solution.

Chapter 3, Section 3.2, Exercises 23-27, page 106. Use the **Growth and Decay Tool**. On the slider for "Decay", pick the value of r to be -.25. Click on the left graph at the initial point (0,18). From the right graph, estimate the half-life. Does your answer match what it should be by an algebraic computation?

Chapter 3, Section 3.2, Logistic Growth, page 101. Use the **Logistic Growth Tool**. For a chosen value of *r*, using the sliders to select values for N_0 and K that satisfy $N_0 < K$, $N_0 > K$, and $N_0 = K$. Describe the changes in population as time increases. What is the equilibrium solution?

Chapter 3, Section 3.4, Example 1, page 116. Use the **Golf Tool**. Using the slider set the air resistance to about .25 and the initial speed of the ball to 200 ft/sec. Now vary the initial angle and observe the flight of the ball by clicking on the left graph. What is the approximate angle that gives the maximum distance for the ball? Does this model accurately depict what happens in reality? What is the maximum range if the air resistance is 0?

Chapter Four

Chapter 4, Sections 4.1 and 4.11, Example 1, page 231. Use the **Simple Harmonic Oscillator Tool**. Set initial conditions for the position and velocity by clicking on the phase plane. In the graphs of the time series, make sure that the position x, velocity $\dot{x}$, and acceleration $\ddot{x}$ are all being graphed simultaneously. Describe how the position and velocity are related. For what position is the velocity a maximum or a minimum? For what velocity is the acceleration a maximum?

Chapter 4, Section 4.11, Example 2, page 235. Use the **Mass and Spring Tool**. Use the sliders pick initial values for *m* and *k* so that b_c is approximately 2. Then use the slider for *b* to pick three values for *b*: one much less than b_c, one equal to b_c, and one much greater than b_c. Compare the graphs and the motion of the spring-mass system. Which case is oscillating, which case is overdamped, and which case is critically damped?

Chapter 4, Section 4.11, Example 2, page 235. Use the **Critical Damping Tool**. Using the slider, pick at least three values for *b*: one less than the critical value (the underdamped case), the critical value itself (the critically damped case), and one greater than the critical value (the overdamped case). Notice the difference of the roots of the auxiliary equation and the difference in the graphs of the solutions.

Chapter 4, Section 4.12, pages 240-246. Use the **Damped Forced Vibrations Tool**. Using the sliders first pick *F* to be 0 and pick *b* to be a nonzero value. Click on the graph of the time series graph at a rather large initial value. Observe the graphs of the solution and make sure by your selections that there is some visible oscillation. What is the steady-state solution in this case? Now repeat the experiment for a nonzero value of *F*. What does the steady-state solution look like now?

Chapter 4, Group Project 4, page 255. Use the **Pendulums Tool**. Click on the "Linear" Button and then on any point in the phase plane. Observe the graph. Now click on the "Simple Nonlinear" Button and on the same point in the phase plane. Observe the graph and any differences with the previous graph. Repeat the experiment with various other points. At what initial values are the graphs most similar and when are they the most dissimilar?

Chapter 4, Frequency Response Curve, Figure 4.34, page 243. Use the **Vibrations: Amplitude Response Tool**. Observe how the frequency response curve changes for various values of *b*. On the slider for *b* set its value to be about .25. Pick various values for *w* on the slider and notice that a horizontal line at the value of *A* is being drawn on the time series graph. Set *w* to be 1 and then successively larger values. For each value of *w* chosen, observe the amplitude of the solution after the transient part becomes negligible. For a small value of *b* like .25, for which value of *w* is the system at resonance?

Chapter 4, Forced Vibrations, page 240. Use the **Vibrations: Input/Output Tool**. Using the sliders, pick various values for *b* and *w*. Observe the graphs on the left - the graph of the solution *x(t)* vs. the applied forcing term $\cos w(t)$. The gray graph is the transient part of the solution and the red is the long-term forced response. First set *w* to be 1 and chose respectively small and large values of *b*. What do the graphs on the left look like? Notice in the time series graphs what happens to the graph of *x(t)* (respectively *F(t)*) when the graph of *F(t)* (respectively *x(t)*) crosses the *t*-axis. Now set *b* equal to a fixed value, and let *w* vary from small to large values. What happen to the ellipses on the left and the curves on the right?

Chapter Five

Chapter 5, Section 5.2, pages 262. Use the **Phase Plane Drawing Tool**. Draw a curve in the *xy*-plane and observe the resulting curves in the *xt*-plane and the *ty*-plane. Experiment by drawing semicircles or spirals and vary the speed at which you draw your curves. Make sure that you notice the relationships between all four graphs.

Chapter 5, Section 5.2, Examples 1 and 2, pages 266-267. Use the **Vector Fields Tool**. From the menu of systems select several different systems. With the vector options selected in the Drawing Mode, draw a number of vectors to get a sense of the flow of the trajectories. Also click the Draw Field icon to draw the direction field. Click the solutions option in the Drawing Mode to draw several solutions and see how they fit into the direction and vector field. From the phase plane, see if you can spot equilibrium points. What do solutions look like near equilibrium points in the phase plane?

Chapter 5, Section 5.2, Example 3, pages 267. Use the **Romeo and Juliet Tool**. In this particular tool, *x* denotes Romeo's love for Juliet and *y* denotes Juliet's love for Romeo. Their love for each other changes according to a system like that of Example 3. Vary the parameters, taking first of all *h* and *k* equal to 0, and compare the various phase planes and trajectories. Can you find values for the matrix for which their love for each other goes in an endless cycle?

Chapter 5, Section 5.5, Table 5.2, page 299. Use the **Series Circuits Tool**. First set *R* and *A* equal to 0. Pick initial values for *q* and $\dot{q}$ by clicking on the graph. Do the graphs and behavior remind you of a mass-spring system? To what is *1/C* analogous? To what is *L* analogous? What is the natural frequency of the system? Now pick *A* to be a nonzero value and pick *w* to be a value close to or equal to the natural frequency. Observe what happens to the current and the charge.

Chapter 5, Section 5.7, Poincare′ Sections, page 316. Use the **Force Damped Pendulum: Nine Sections Tool**. Set the parameters *b* = .5 and *w* = .67 on the sliders. Now sequentially set the values for *F* to be 1.50, 1.35, 1.45, and 1.47. Notice how the Poincare′ Sections depend on the sampling time. In each case, after you clear the transients, determine if any of the pictures are identical.

Chapter 5, Section 5.7, Equation(8), page 321. Use the **Forced Damped Pendulum Tool**. Set the parameters *b* = .5 and *w* = .67 on the sliders. Now sequentially set the values for *F* to be 1.50, 1.35, 1.45, and 1.47. In each case clear the transients and observe the changes of behavior. Do any solutions appear to be periodic? Do any appear to be chaotic?

Chapter 5, Group Project D, page 331. Use the **Lotka-Volterra Tool**. In order to get the system referred to in this project, set the values on the sliders all equal to 1. Look at the resulting phase plane. Click on the only critical points and see the resulting behavior of the two variables on the time series graphs. Click anywhere else on the phase plane and observe the trajectories. Is there symmetry about the line *y* = *x* (in this case *P* = *H*)?

Chapter Six

Chapter 6, Section 6.1, Theorem 1, page 339. Use the **Mass and Spring Tool**. Click on the phase plane graph on the left at any point away from the origin. (We do this so that you can see the resulting graph.) Do any of the resulting graphs intersect in the phase plane? Can they according to Theorem 1?

Chapter 6, Section 6.1, pages 338-345. Use the **Matrix Machine Tool**. Set the numerical values of the matrix *A* so that you have a nonzero matrix with a zero determinant. By moving the pointer in the left graph, now find a nonzero vector *v* so that *Av* is zero. Is this possible if the determinant of *A* would be *0*?

Chapter 6, Section 6.1, pages 338-345. Use the **Eigen Engine Tool**. Set arbitrary values in the matrix *A*. Move the cursor in the left graph until it freezes and the input vector *v* is nonzero. Read off the value of λ and by hand compute the value of the determinant of *A* - λ *I.* What kind of answer do you get approximately? What should the answer be exactly?

Chapter Seven

Chapter 7, Laplace Transform Definition, page 373. Use the **Laplace: Definition Tool**. First pick various values of *s* on the slider on the right. Notice what happens to the graph of $e^{-st}\sin t$ as the values of *s* change. On the bottom left is the graph of *I(s,T)*, whose limit we take as $T->\times$ to get the Laplace Transform. As the values of *T* are changed on this graph, what is the relationship of *I(s,T)* to the graph of the function $e^{-st}\sin t$?

Chapter 7, Laplace Transforms and Inverse Laplace Transforms, pages 573-596. Use the **Laplace: Transformer Tool**. Click on the "Laplace" button and "Set Function" button to select a few functions to "transform" by clicking on the "*L*" button. What do you observe about the transformed function *F(s)* as *s* gets large? Now click on the "Inverse" button and the "Set Function" button to select a few functions to "inverse transform" by clicking on the "L^{-1}" button.

Chapter 7, Laplace Transform of the Derivative, page 383. Use the **Laplace: Derivative Tool**. According to Theorem 4 (page 383), $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$. See this demonstrated for $\cos(2t)$ by clicking on the clicking on the "differentiate" button and then the "L" button and observing the resulting graph. Now clear, and click first on the "L" button and then the "Derivative Theorem" button. Do you get the same graph?

Chapter 7, Section 7.3, Exercises 1-20, pages 386-387. Use the **Laplace: Vibrations and Poles Tool**. This tool finds the Laplace transform of $\ddot{x}+2b\dot{x}+x=0$ for various values of *b*. Let *b* vary from 0 to various values above the critical damping value b_c = 1, the critical damping value itself, and above the critical damping value. Observe that the auxiliary equation is the same as the denominator of the Laplace transform of *x.* Observe the location of the roots of this equation in the top graph, the graph of the solution, and the graph of the Laplace transform of the solution in the lower graph.

Chapter 7, Review Problems 19-24, page 441. Use the **Laplace: Solver Tool**. On the bottom of the template, you will find several choices for forcing function *f(t)* for the equation $\ddot{x}+x=f(t)$ and two choices for initial values for *x(0)*. Select various choices and click on the buttons for "L", "solve", and "L^{-1}", in that order. Observe the resultant change on the solution *x(t)* in the lower left-hand graph.

Chapter Eight

Chapter 8, Section 8.1, Example 1, page 449. Use the three tools **Maclaurin Series: e^t, Cosine, and Sine**. For each of these tools, corresponding respectively to the functions $e^t, \cos(t)$, and $\sin(t)$, select different numbers of terms and see how close the series approximations are. Investigate the fact that, near 0, the error is well approximated by the absolute value of the first deleted term.

Chapter 8, Group Project B, page 524. Use the **Airy's Series: Cosine Tool**. The first five terms of the power series expansion about 0 are shown and the graph in the interval [0,5]. By clicking on the icons, you can see the graphs of the polynomial approximations for 1 to 5 terms. Notice how the approximations become more accurate with an increasing number of terms.

Chapter 8, Group Project B, page 524. Use the **Airy's Equation Tool**. Click on Airy Sine and Airy Cosine respectively. Compare the resulting phase planes and the time series graphs. In each case observe the behavior of the general solution to Airy's equations when $t > 0$ and when $t < 0$.

Chapter 8, Exercise 38, Section 8.8, page 519. Use the **Chebyshev's Equation Tool**. Click on the Chebyshev Cosine. On the slider for n, pick various values for n starting at $n = 0$ and including integer values from 1 to 8. See if the Chebyshev polynomials that you determined in Exercise 38 coincide with the polynomials given in the template of this tool. What are the values of all the Chebyshev polynomials for $n = 1, 2,...,8$ at the point $t = 0$?

Chapter Nine

Chapter 9, Section 9.3, Theorem 1, page 543. Use the **Matrix Machine Tool**. By adjusting the numerical values for the entries of the matrix $\boldsymbol{A}$, find a nonzero matrix whose determinant is zero. By moving the pointer on the graph, locate vectors $\mathbf{x}$ so that $\boldsymbol{A}\mathbf{x} = \mathbf{0}$. What is characteristic about all the vectors that you find? Next find a matrix whose determinant is nonzero. Again locate all the vectors $\mathbf{x}$ such that $\boldsymbol{A}\mathbf{x} = \mathbf{0}$. What is characteristic of these vectors?

Chapter 9, Section 9.5, Example 1, page 558. Use the **Eigen Engine Tool**. Set the matrix equal to the one in Example 1. Click on the graph and move the pointer around until it freezes. Read off the eigenvector. Click on the graph and move the pointer around again until you get a second eigenvector. Observe what eigenvalues and eigenvectors mean geometrically. Repeat for other matrices.

Chapter 9, Section 9.5, Example 3, page 561. Use the **Matrix Element Input Tool**. Set the matrix A equal to the one of Example 3. Click in the graph above the matrix. On the graph on the right, do you see the lines through the eigenvalues? For what values of c_1 and c_2 are all the points on these lines solutions? Now click on the black graph at any point. The line that you get represents values of *x(t)* for some values of c_1 and c_2 .

Chapter Ten

Chapter 10, Example 4, Section 10.3, page 619. Use the **Fourier Series: Square Wave Tool**. Notice that the function used is the same as in Example 4. Click on all the different button options to observe the graph of each term as well as how the sum of terms approximate the square wave function. Why does the Fourier series contain only sine functions? By looking at the successive graphs of the sums, can you determine to which function the series converges? (See Example 6 on page 624.)

Chapter 10, Example 5, Section 10.3, page 620. Use the **Fourier Series: Triangle Wave Tool**. Notice that the function used is similar to that used in Example 5. Click on all the different button options to observe the graph of each term as well as how the sum of terms approximate the triangle function. Why does the Fourier series contain only cosine functions? By looking at the successive graphs of the sums, can you see uniform convergence here? (See Theorem 3 on page 625.)

Chapter 10, Exercise 39, Section 10.3, page 630. Use the **Fourier Series: Gibbs Effect Tool**. On the bottom of the template are the graphs of the square wave and its Fourier approximates. The left graph is the magnification of the right graph. Does the maximum error in the Fourier series approximation of the square wave ever go to zero? Does it remain larger than some positive value? Estimate the value.

Chapter Eleven

There are no tools for this chapter.

Chapter Twelve

Chapter 12, Section 12.2, pages 765-777. Use the **Parameter Path Animation Tool**. On the template is the plane of roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha + i\beta$ of the characteristic equation (5, page 766). The regions are as follows: magenta-both r_1 and r_2 are positive and real; cyan-both $r_1 = \alpha + i\beta$ and $r_2 = \alpha + i\beta$ are complex and α is positive; red-both $r_1 = \alpha + i\beta$ and $r_2 = \alpha + i\beta$ are complex and α is negative; green-both r_1 and r_2 are negative and real; blue-both r_1 and r_2 are real but of opposite sign. On the parabola itself the roots are equal. Click on the single > or < and look at how the solutions change when the roots are in the various areas.

Chapter 12, Section 12.2, pages 765-777. Use the **Parameter Plane Input Tool**. On the template is the plane of roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha + i\beta$ of the characteristic equation (5, page 766). The regions are as follows: magenta-both r_1 and r_2 are positive and real; cyan-both $r_1 = \alpha + i\beta$ and $r_2 = \alpha + i\beta$ are complex and α is positive; red-both $r_1 = \alpha + i\beta$ and $r_2 = \alpha + i\beta$ are complex and α is negative; green-both r_1 and r_2 are negative and real; blue-both r_1 and r_2 are real but of opposite sign. On the parabola itself the roots are equal. Click on the plane of the roots. Notice the phase plane and the graph of the characteristic equation on the bottom. Click on the phase plane to see trajectories. In each case, classify the equilibrium point as stable or unstable, proper, improper, a spiral point, or a center.

Chapter 12, Equation (5), Section 12.2, page 767. Use the **Four Animation Paths Tool**. In this tool, you can examine what happens on the borders between region of the plane of roots of equation (5). Along the horizontal axis the coefficient of *r* is 0 and along the vertical axis the constant term is 0. Along the parabola the roots are equal. Pick the various selections on the bottom left and set the animation going. Observe the behavior of trajectories in each case and the classifications that are shown.

Chapter 12, Competing Species, page 783. Use the **Competitive Exclusion Tool**. Observe the affect on the phase plane of changing the parameters on the sliders. What do the two straight lines on the phase plane represent? Where are the equilibrium points? Click on various points on the phase plane and determine what happens to the population as time increases. Read from the screen the coordinates of the equilibrium points.

Chapter Thirteen

There are no tools for this chapter.