Project 5 Tutorial

A. Introduction

In this tutorial you will learn some tools for studying first order differential equations using Maple V. In particular, we will see how to find the general and particular solutions of certain first order equations. In addition we will learn how to plot direction fields for first order scalar equations and systems of equations. Solution curves can also be added to these plots. The following Maple V commands are used in this lesson: diff, int, lhs, rhs, solve, unapply, eval, subs, simplify, dsolve, plot, plots[display], with(DEtools), DEplot, with(linalg), jacobian, multiply, vector, convert

B. Analyzing Systems of Equations

(a) The following example shows how Maple V allows us to study a system of differential equations. Consider the system:

$$\text{(DE1)} \qquad \frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{x}(t)\,\mathrm{y}(t)\,, \quad \frac{\partial}{\partial t}\mathrm{y}(t) = 2 - \mathrm{y}(t)\,.$$

```
> restart:
> with(DEtools): with(linalg): with(plots):
Warning, new definition for norm
Warning, new definition for trace
> de1a := diff(x(t),t)=x(t)*y(t);
```

$$de1a := \frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{x}(t)\,\mathrm{y}(t)$$

```
> de1b := diff(y(t),t)=2-y(t);
```

$$de1b := \frac{\partial}{\partial t}\mathrm{y}(t) = 2 - \mathrm{y}(t)$$

In this example we use the "DEtools" procedure "DEplot". The direction field for the system is obtained with the following command.

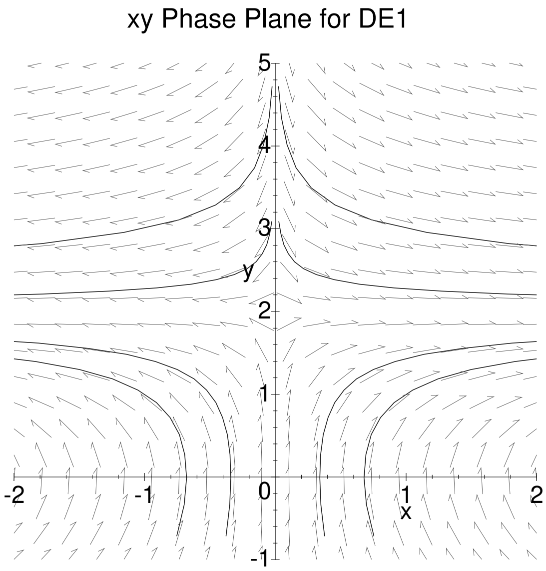
```
> P1 := DEplot({de1a,de1b},[x(t),y(t)],t=0..1,x=-2..2, y=-1..5, arrows=SMALL):
```

The next two commands produce a plot that creates some solution curves. The t interval [-1,2] is used so that the curves will go both forward and backward from the initial points.

```
> P2 := DEplot({de1a,de1b},[x(t),y(t)],t=-1..2, {[x(0)=-1,y(0)=1],
  [x(0)=-0.5,y(0)=1], [x(0)=0.5,y(0)=1], [x(0)=1,y(0)=1]}, arrows=NONE,
  method=rkf45, linecolor=blue):
> P3 := DEplot({de1a,de1b},[x(t),y(t)],t=-1..2, {[x(0)=0.4,y(0)=2.4],
  [x(0)=-0.4,y(0)=2.4], [x(0)=-1,y(0)=3], [x(0)=1,y(0)=3]}, arrows=NONE,
  method=rkf45, linecolor=blue):
```

The next command plots the solution curves along with the direction field.

```
> display({P1,P2,P3}, view=[-2..2,-1..5], title=`xy Phase Plane for DE1`,
  font=[HELVETICA,BOLD,12], titlefont=[HELVETICA,BOLD,14]);
```



Observe that this plot suggests that the point (0,2) is an equilibrium point. Let us now analyze the linearization of the system near this point. The jacobian matrix evaluated at the point (0,2) represents the "derivative" of the right hand side, [xy, 2-y], of the system.

```
> J := jacobian([x*y,2-y],[x,y]);
```

$$J := \begin{bmatrix} y & x \\ 0 & -1 \end{bmatrix}$$

```
> J02 := subs(x=0,y=2,eval(J));
```

$$J02 := \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

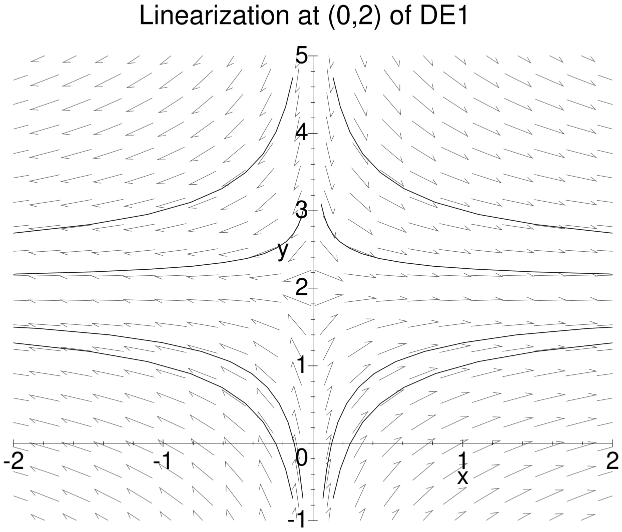
The linearization of the system in a coordinate system (u,v) centered at x=0 and y=2 is given by multiplying the latter matrix by the vector [u,v]. The linearization of the system in the (x,y) coordinate system is obtained by multiplying J02 by the vector [x,y-2].

```
> A := multiply(J02,vector([x,y-2]));
```

$$A := [2\,x, 2 - y]$$

We now obtain solution curves and the direction field for the linearized system.

```
> P4 := DEplot({D(x)(t)=2*x,D(y)(t)=2-y}, [x(t),y(t)], t=0..1,
  x=-2..2,y=-1..5, arrows=SMALL):
> P5 := DEplot({D(x)(t)=2*x,D(y)(t)=2-y},[x(t),y(t)],t=-1..2,
  [[x(0)=-1,y(0)=1], [x(0)=-0.5,y(0)=1], [x(0)=0.5,y(0)=1], [x(0)=1,y(0)=1]],
  arrows=NONE, method=rkf45, linecolor=blue):
> P6 := DEplot({D(x)(t)=2*x,D(y)(t)=2-y}, [x(t),y(t)], t=-1..2,
  [[x(0)=0.4,y(0)=2.4], [x(0)=-0.4,y(0)=2.4], [x(0)=-1,y(0)=3],
  [x(0)=1,y(0)=3]], arrows=NONE, method=rkf45, linecolor=blue):
> display({P4,P5,P6},view=[-2..2, -1..5], title=`Linearization at (0,2) of
  DE1`, font=[HELVETICA,BOLD,12], titlefont=[HELVETICA,BOLD,14]);
```



Observe that the qualitative behavior of the phase plane of the linearization seems to agree with that of the nonlinear system near the equilibrium point (0,2).

(b) We now consider a another system of equations:

$$\text{(DE2)} \qquad \frac{\partial}{\partial t}\mathrm{x}(t) = -\mathrm{x}(t) + \frac{\mathrm{x}(t)\,\mathrm{y}(t)}{100}, \quad \frac{\partial}{\partial t}\mathrm{y}(t) = 2\,\mathrm{y}(t) - 2\,\mathrm{x}(t)\,\mathrm{y}(t),$$

and this time we will use vector notation for the system.

```
> f:=(x,y)->-x+x*y/100: g:=(x,y)->2*y-2*x*y/25:
> DE := [diff(x(t),t)= f(x(t),y(t)), diff(y(t),t) = g(x(t),y(t))];
```

$$DE := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = -\mathrm{x}(t) + \frac{1}{100}\,\mathrm{x}(t)\,\mathrm{y}(t), \frac{\partial}{\partial t}\mathrm{y}(t) = 2\,\mathrm{y}(t) - \frac{2}{25}\,\mathrm{x}(t)\,\mathrm{y}(t)\right]$$

Let us determine the equilibrium points for this system. The equilibrium points are those points where the right hand side of each equation vanishes. The "solve" command will do this.

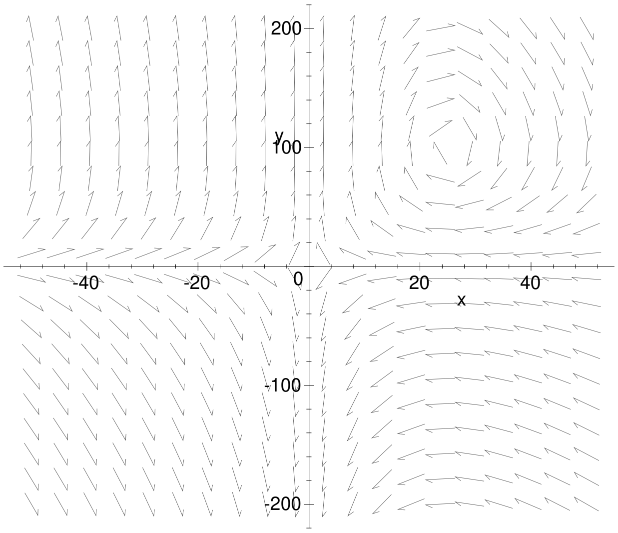
```
> solve({f(x,y)=0,g(x,y)=0},{x,y});
```

$$\{x = 0, y = 0\}, \{y = 100, x = 25\}$$

Thus we conclude that equilibrium points for this system are the points (0,0) and (25,100). Let us now plot the direction field for the system in the region x in [-60,60], and y in [-300,300].

```
> G1 := DEplot({DE[1],DE[2]},[x(t),y(t)],t=0..1, x=-50..50, y=-200..200,
  color=magenta):
> display(G1, title=`Direction Field of System DE`, font=[HELVETICA,BOLD,10],
  titlefont=[HELVETICA,BOLD,14]);
```

Direction Field of System DE



There are a number of things to observe in the preceding plot. Note that the field becomes more interesting near the equilibrium points. Also note that the field is parallel to the coordinate axes at all points along the coordinate axes. This means that each coordinate axis is invariant relative to the flow of the system, in the sense that once a solution curve is on an axis it remains on that axis for all time. This also implies that each quadrant is invariant. Thus, for example, a solution curve which is initially in the first quadrant will remain in that quadrant. This is important in biological models where x and y may denote populations of certain species and negative populations make no sense. Let us now find the linear approximations to the system at the two equilibrium points. First we find the linearization at the point (0,0).

```
> J := jacobian([f(x,y),g(x,y)],[x,y]);
```

$$J := \begin{bmatrix} -1 + \frac{1}{100}y & \frac{1}{100}x \\ -\frac{2}{25}y & 2 - \frac{2}{25}x \end{bmatrix}$$

```
> J00 := subs(x=0,y=0,eval(J));
```

$$J00 := \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> eigenvects(J00);
```

$$[-1, 1, \{[1, 0]\}], [2, 1, \{[0, 1]\}]$$

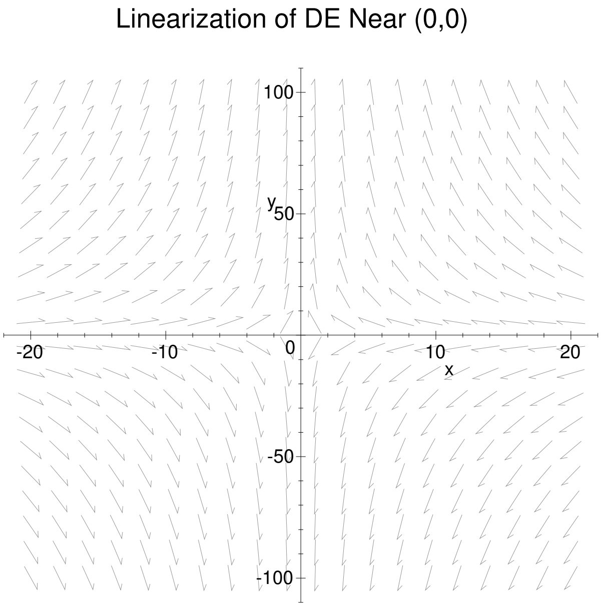
This means that the point (0,0) is a saddle point. The linear approximation of the system near (0,0) is the vector field:

```
> V[1] := multiply(J00,[x,y]);
```

$$V_1 := [-x, 2\,y]$$

The direction field for this linearization is generated with the following command.

```
> DEplot({D(x)(t)=-x,D(y)(t)=2*y},[x(t),y(t)],t=0..1, x=-20..20,y=-100..100,
  color=green, title=`Linearization of DE Near (0,0)`,
  font=[HELVETICA,BOLD,10], titlefont=[HELVETICA,BOLD,14]);
```



Observe that, as in our previous examples, the qualitative behavior of the linear approximation of the system at the equilibrium point (0,0) is the same as that of the nonlinear system near (0,0). We now investigate the behavior of the linearization at the point (25,100).

```
> J25100 := subs(x=25,y=100,eval(J));
```

$$J25100 := \begin{bmatrix} 0 & \frac{1}{4} \\ -8 & 0 \end{bmatrix}$$

```
> eigenvects(J25100);
```

$$\left[I\sqrt{2},\, 1,\, \left\{\left[-\frac{1}{8}I\sqrt{2},\, 1\right]\right\}\right], \left[-I\sqrt{2},\, 1,\, \left\{\left[\frac{1}{8}I\sqrt{2},\, 1\right]\right\}\right]$$

The critical point (25,100) appears to be a center.

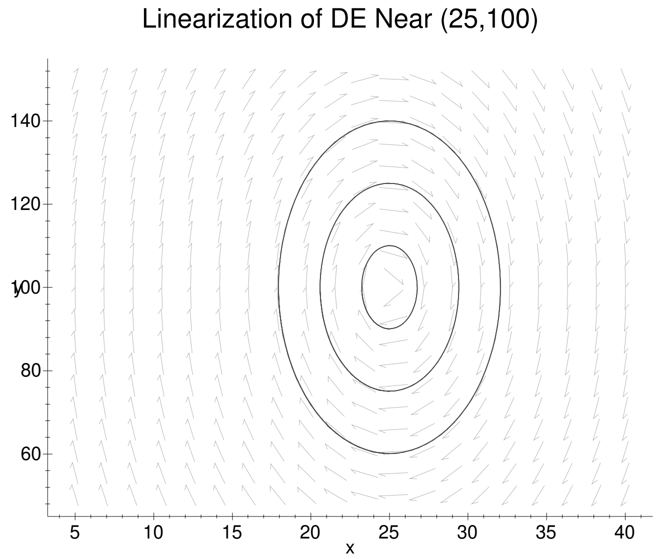
```
> V[2] := multiply(J25100,[x-25,y-100]);
```

$$V_2 := \left[-25 + \frac{1}{4}y,\, 200 - 8x\right]$$

```
> L1 := DEplot({D(x)(t)=V[2][1],D(y)(t)=V[2][2]}, [x(t),y(t)],
  t=0..1,x=5..40,y= 50..150, arrows=SMALL, color=aquamarine):
```

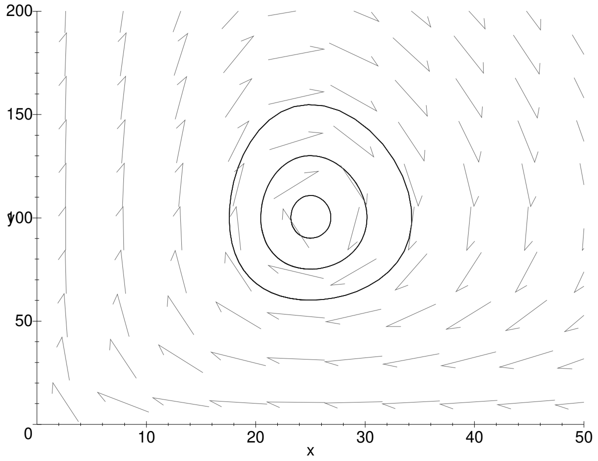
For illustration purposes we will now plot this direction field with a few solution curves of the linear system.

```
> L2 := DEplot({D(x)(t)=V[2][1],D(y)(t)=V[2][2]}, [x(t),y(t)], t=0..10,
  [[x(0)=25,y(0)=60], [x(0)=25,y(0)=75], [x(0)=25,y(0)=90]], x=5..40,
  y=50..150, stepsize=0.1, arrows=NONE, linecolor=maroon):
> display({L1,L2}, title=`Linearization of DE Near (25,100)`,
  font=[HELVETICA,BOLD,10], titlefont=[HELVETICA,BOLD,14]);
```



As we might now be able to predict, the behavior of the linearized vector field is that of the original nonlinear system near the equilbrium point (25,100). In this case the solution curves form three concentric closed curves centered at the equilibrium point. Let us now compare the latter plot with solution curves with the same initial conditions for the original nonlinear system.

```
> G2 := DEplot({DE[1],DE[2]},[x(t),y(t)],t=0..10, [[x(0)=25,y(0)=60],
  [x(0)=25,y(0)=75], [x(0)=25,y(0)=90]], x=-50..50, y=-200..200, stepsize=0.1,
  method=rkf45, linecolor=blue, arrows = NONE):
> display({G1,G2},view = [0..50,0..200]);
```

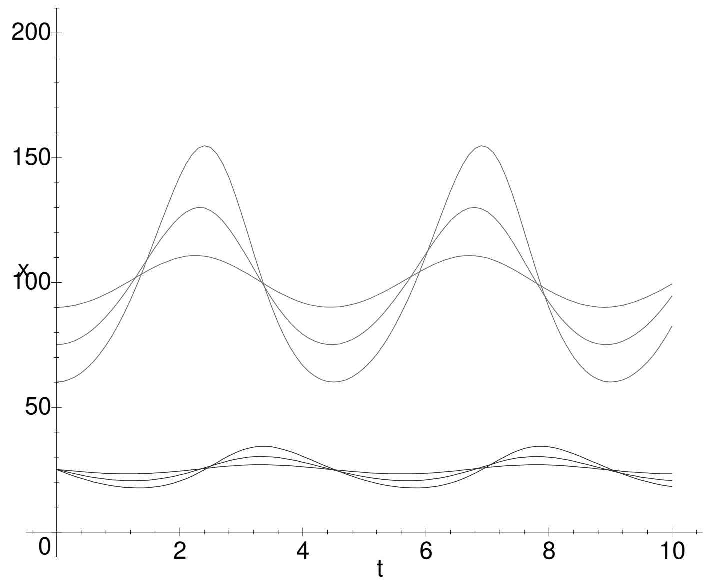


Notice that the solution curves of the nonlinear system that are near the equibrium point (25,100) are like the solution curves for the linear approximation at that point. We will now plot a time domain plot for the same three solutions of the nonlinear system. We use the "scene" option to generate a plot of t vs x(t) in color "navy" and a plot of t vs y(t) in color "magenta." We then plot both on the same axes by using the DISPLAY command.

```
> TX := DEplot({DE[1],DE[2]},[x(t),y(t)],t=0..10, [[x(0)=25,y(0)=60],
  [x(0)=25,y(0)=75], [x(0)=25,y(0)=90]], method=rkf45, arrows=NONE,
  x=0..60,y=0..200, scene=[t,x], stepsize=0.1, linecolor=navy):
> TY := DEplot({DE[1],DE[2]},[x(t),y(t)],t=0..10, [[x(0)=25,y(0)=60],
  [x(0)=25,y(0)=75], [x(0)=25,y(0)=90]], method=rkf45, arrows=NONE,
  x=0..60,y=0..200, scene=[t,y], stepsize=0.1, linecolor=magenta):
```

```
> XY := DEplot({DE[1],DE[2]},[x(t),y(t)],t=0..10, [[x(0)=25,y(0)=60],
  [x(0)=25,y(0)=75], [x(0)=25,y(0)=90]], method=rkf45, arrows=NONE,
  x=0..50,y=30..180, scene=[x,y], stepsize=0.1, linecolor=magenta):
> display({TX,TY},font=[HELVETICA,BOLD,12], title=`Plot of t vs x(t) and
  y(t)`,titlefont=[HELVETICA,BOLD,14]);
```

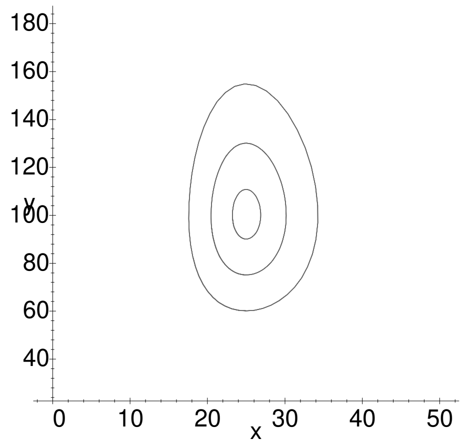
Plot of t vs x(t) and y(t)



Note that the t vs y(t) curves have larger amplitude than the t vs x(t) curves. Also observe that both solutions seem to be periodic with period around 4.5.

```
> display(XY,font=[HELVETICA,BOLD,12], title=`Plot x(t) vs
  y(t)`,titlefont=[HELVETICA,BOLD,14]);
```

Plot x(t) vs y(t)




```
>
```