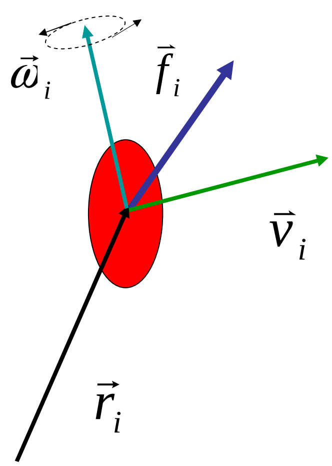
Mechanics II

< draft >

Geometry of motion in 3-dimensional space

The motion of an object in space can be decomposed into a translational motion of its center of mass and a rotational motion about its center of mass:

$$m \frac{d^2 \vec{r}_i}{dt^2} = \vec{f}_i$$

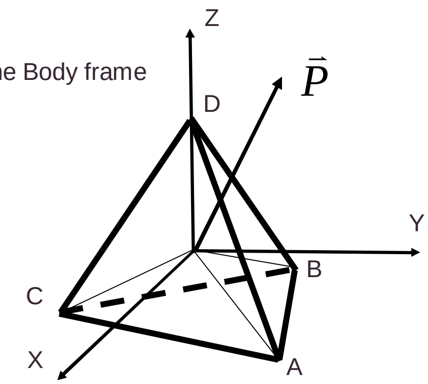


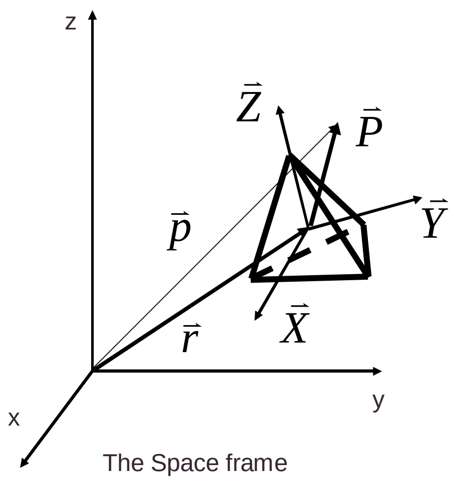
(1) Translation of the center of mass obeys Newton's equation

(2) Rotation about the center of mass obeys Euler's equations

$$\vec{p} = \vec{r} + Q\vec{P}$$

The Body frame





The Space frame

$$Q = \begin{pmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{pmatrix}$$

X,Y,Z expressed in Space frame provide
the three columns of the rotation matrix

To describe the motion of an object in space, it is convenient to derive an expression for converting from the space to the body frames and back: for vectors other than the position vector and velocity, (i.e. angular velocity and momentum, torque, force), the transformation is performed via the instantaneous rotation matrix, Q(t):

$$\vec{u}(t) = Q(t)\vec{U}(t) \Rightarrow \vec{U}(t) = Q^T(t)\vec{u}(t)$$

differentiating wrt. time:

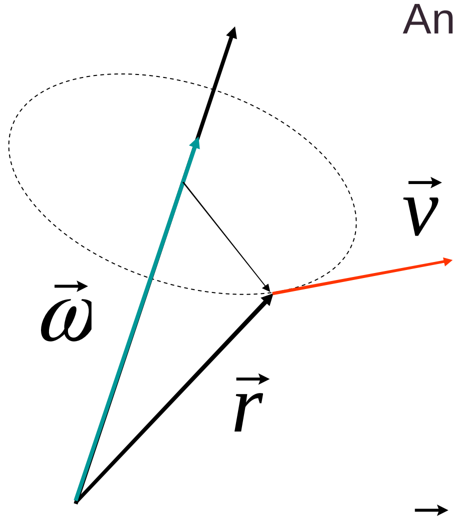
$$\frac{d\vec{u}(t)}{dt} = \frac{dQ(t)}{dt}\vec{U}(t) + Q(t)\frac{d\vec{U}(t)}{dt}$$

We will now discuss how to think of the terms in this equation
in an intuitive way. We will need to introduce some new concepts!

Concepts and definitions:

(1) Angular velocity
(2) Angular momentum
(3) Moments of inertia
(4) The Inertia matrix A
(5) The relationship between angular velocity and angular momentum
(6) Principal axes of inertia
(the eigenvalues and eigenvectors of the matrix A)
(7) Torque and the change in angular momentum

Angular velocity



$$\vec{v} = \vec{\omega} \times \vec{r}$$

It can be shown that the Body-to-Space matrix Q(t) and the angular velocity in the body are related by

$$QQ^T = I \Rightarrow \frac{dQ(t)}{dt}Q^T = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}$$

The matrix Q is given in Appendix A in terms of the components of the rotation quaternion q, based on the evolving axis and angle of rotation

Introducing quaternions, and from the relation of Q(t) to q(t) given in Appendix A, it can be shown that:

$$\frac{dq}{dt} = \frac{1}{2}q\Omega;$$

$$q = [q_0, q_1, q_2, q_3]; \Omega = [0, \Omega_1, \Omega_2, \Omega_3]$$

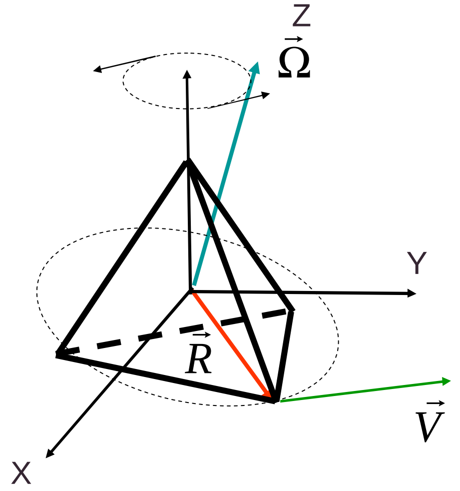
$$\vec{\omega} = Q\vec{\Omega} \Rightarrow \omega = q\Omega q^c \Rightarrow \Omega = q^c \omega q$$

Since $qq^c = 1 \Rightarrow \frac{dq}{dt}q^c = -q\frac{dq^c}{dt}$

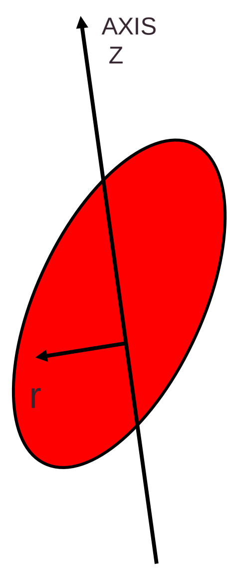
$$\Rightarrow \frac{dq^c}{dt} = q^c \frac{dq}{dt} q^c \Rightarrow \frac{dq^c}{dt} = -\frac{1}{2}\Omega q^c$$

The angular velocity
in the body frame

$$\vec{V} = \vec{\Omega} \times \vec{R}$$



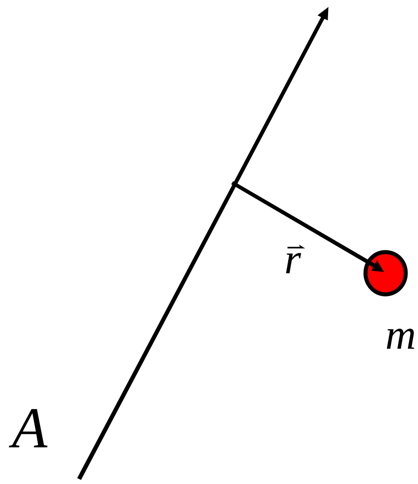
(1) Moments of inertia



$$I_z = \iiint_{body} \rho(\vec{r}) r^2 dV$$

The moment of inertia of a body about an axis is given by the density times the square of the distance from the axis integrated over the body

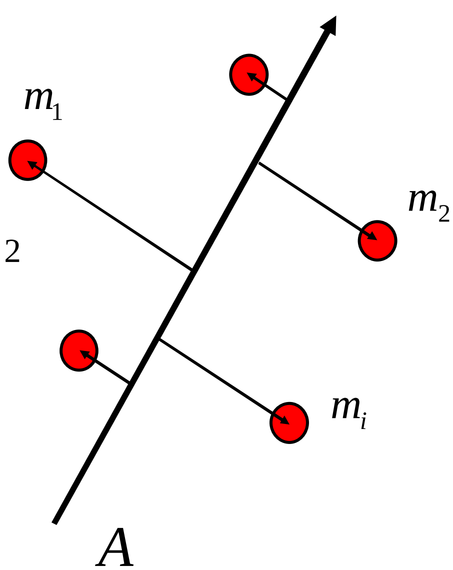
Example: Point mass m



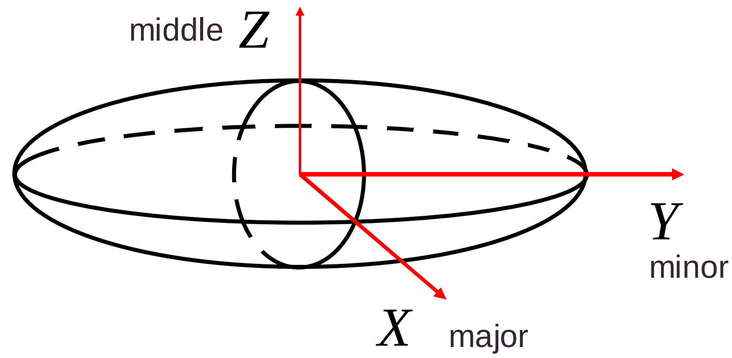
$$I_A = mr^2$$

A collection of points:

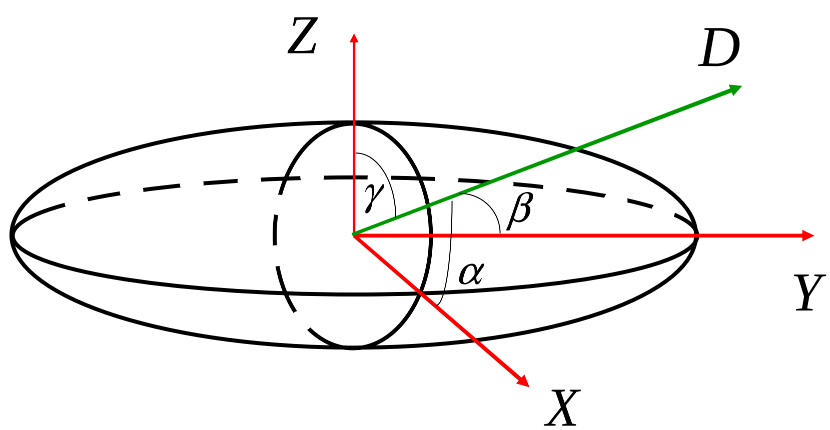
$$I_A = \sum_i m_i r_i^2$$



FACT: each body can be completely characterized, for the purpose of studying its motion, by three axes. These axes are mutually orthogonal, and they are called principal axes of inertia.

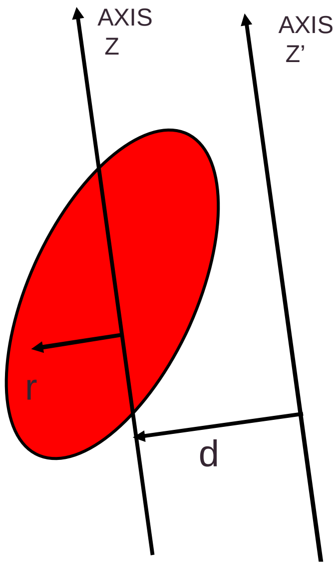


The principal axes of inertia of a football



Specifying the moments of inertia about the principal axes, allows us to find easily the moment of inertia about any axis:

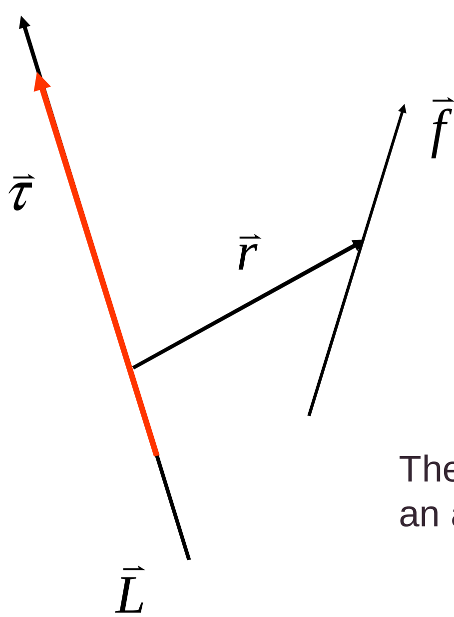
$$I_D = I_{XX}\cos^2\alpha + I_{YY}\cos^2\beta + I_{ZZ}\cos^2\gamma$$



$$I_{Z'} = I_Z + md^2$$

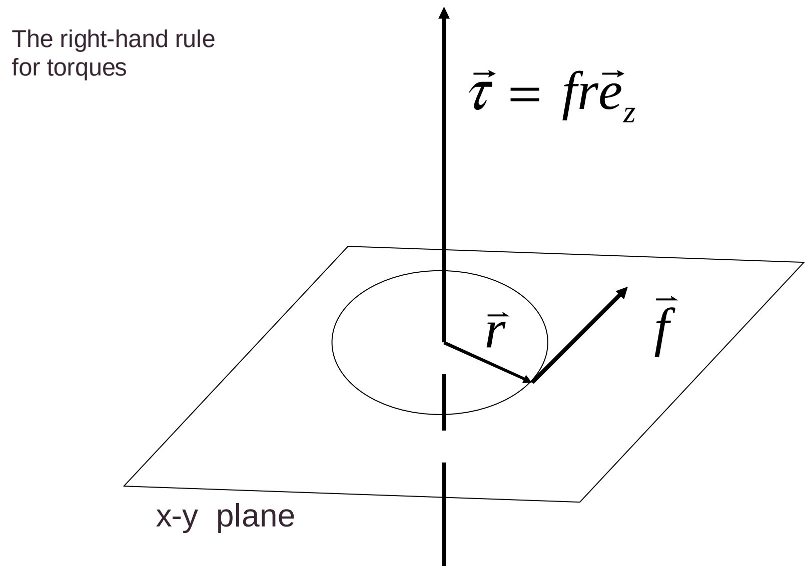
The parallel axes theorem: how to compute moments about shifted axes

Torque

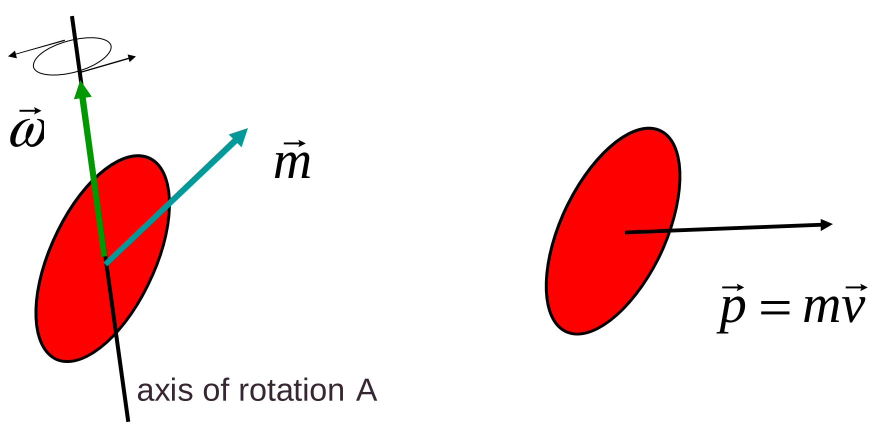


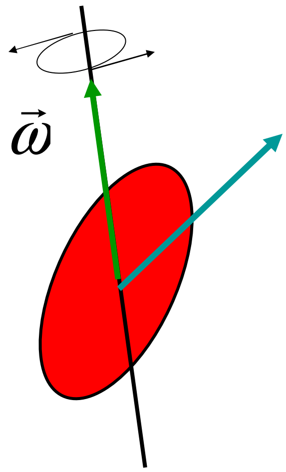
$$\vec{\tau} = \vec{r} \times \vec{f}$$

The torque of a force about an axis in space



The angular momentum plays a similar role for rotational motion as the linear momentum does for rectilinear motion: however, unlike the linear momentum which points in the direction of the velocity, the angular momentum does not in general point in the same direction as the angular velocity





axis of rotation L

$$\vec{m} = \sum_{body} m_i \vec{r} \times \vec{v}_i$$

$$= \sum_{body} m_i \vec{r} \times \left(\vec{\omega} \times \vec{r}_i\right)$$

$$= A\vec{\omega}$$

The law of motion for a system rotating about its center of mass relates the externally applied torques to the angular momentum:

$$\left(\frac{d\vec{m}}{dt}\right)_{space} = \vec{\tau}$$

$$\left(\frac{d\vec{m}}{dt}\right)_{space} = \left(\frac{d\vec{M}}{dt}\right)_{body} + \vec{\Omega} \times \vec{m}$$

The moment-of-inertia matrix gives the connection between angular velocity and angular momentum:

$$\vec{M} = \overleftrightarrow{A}\vec{\Omega}$$

If the coordinate system is based on the principal axes of inertia, the moment of inertia matrix assumes a simple (diagonal) form:

$$\vec{M} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \vec{\Omega}$$

$$\left(\frac{d\vec{m}}{dt}\right)_{space} = \vec{\tau} \qquad (1)$$

$$\vec{m} = Q\vec{M} \Leftrightarrow \vec{M} = Q^T \vec{m} \qquad (2)$$

$$\vec{M} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \vec{\Omega} \qquad (3)$$

$$\frac{dq}{dt} = \frac{1}{2} q\Omega; \qquad (4)$$

$$q = [q_0, q_1, q_2, q_3]; \Omega = [0, \Omega_1, \Omega_2, \Omega_3]$$

The solution of (1-4) gives the rotation of the body at all times. To begin, we need a starting orientation and angular velocity; then we need a way of calculating the externally applied torques at all times. We start with a discussion of the motion of a body absent center-of-mass motion and external torques:

$$\vec{m} = Q\vec{M} \Leftrightarrow m = qMq^c$$

$$0 = \frac{dm}{dt} = \frac{dq}{dt}Mq^c + q\frac{dM}{dt}q^c + qM\frac{dq^c}{dt}$$

$$\frac{dq}{dt} = \frac{1}{2}q\Omega \qquad \frac{dq^c}{dt} = -\frac{1}{2}\Omega q^c$$

$$\frac{dq}{dt}Mq^c + q\frac{dM}{dt}q^c + qM\frac{dq^c}{dt} = 0 \Rightarrow$$

$$\frac{1}{2}q(\Omega M)q^c + q\frac{dM}{dt}q^c - \frac{1}{2}q(M\Omega)q^c = 0 \Rightarrow$$

$$q\left(\frac{1}{2}\Omega M + \frac{dM}{dt} - \frac{1}{2}M\Omega\right)q^c = 0 \Rightarrow$$

$$\frac{dM}{dt} + \frac{1}{2}(\Omega M - M\Omega) = 0$$

Rewritting the last equation in terms of vectors:

$$\frac{dM}{dt} + \frac{1}{2}(\Omega M - M\Omega) = 0$$

$$\Omega M - M\Omega = [-\vec{M}\cdot\vec{\Omega}, \vec{\Omega}\times\vec{M}] - \left[-\vec{M}\cdot\vec{\Omega}, \vec{M}\times\vec{\Omega}\right]$$
$$= 2\vec{\Omega}\times\vec{M}$$

So, finally:

$$\frac{dM}{dt} + \vec{\Omega}\times\vec{M} = 0$$

$$\frac{dM}{dt} + \vec{\Omega}\times\vec{M} = 0 \qquad \vec{\Omega} = A^{-1}\vec{M} = \begin{pmatrix} 1/I_1 & 0 & 0 \\ 0 & 1/I_2 & 0 \\ 0 & 0 & 1/I_3 \end{pmatrix}$$

$$\frac{dM_1}{dt} = \Omega_3 M_2 - \Omega_2 M_3 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) M_3 M_2$$

$$\frac{dM_2}{dt} = \Omega_1 M_3 - \Omega_3 M_1 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) M_1 M_3$$

$$\frac{dM_3}{dt} = \Omega_2 M_1 - \Omega_1 M_2 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) M_2 M_1$$

$$\frac{dM_1}{dt} = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) M_3 M_2$$

$$\frac{dM_2}{dt} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) M_1 M_3$$

Euler equations

$$\frac{dM_3}{dt} = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) M_2 M_1$$

$$\vec{m} = Q\vec{M} \Leftrightarrow \vec{M} = Q^T \vec{m}$$

$$\frac{dq}{dt} = \frac{1}{2} q\Omega;$$

$$\vec{\Omega} = A^{-1}\vec{M} = \begin{pmatrix} 1/I_1 & 0 & 0 \\ 0 & 1/I_2 & 0 \\ 0 & 0 & 1/I_3 \end{pmatrix}$$

APPENDIX A

Quaternion operations: multiplication

$$u = [u_0, \vec{u}]$$

$$v = [v_0, \vec{v}]$$

$$uv = [u_0 v_0 - \vec{u}\cdot\vec{v}, u_0\vec{v} + v_0\vec{u} + \vec{u}\times\vec{v}]$$

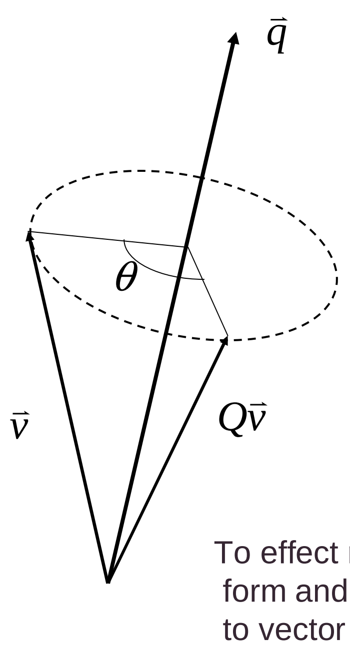
multiplication by a quaternion is a linear operation,

and it can be represented by a matrix:

$$uv = \left[u_0 v_0 - \vec{u}\cdot\vec{v}, u_0\vec{v} + v_0\vec{u} + \vec{u}\times\vec{v}\right]$$

$$uv = A_L(u)v = \begin{pmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & u_3 & -u_2 \\ u_2 & -u_3 & u_0 & u_1 \\ u_3 & u_2 & -u_1 & u_0 \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$vu = A_R(u)v = \begin{pmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & -u_3 & u_2 \\ u_2 & u_3 & u_0 & -u_1 \\ u_3 & -u_2 & u_1 & u_0 \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$



$\vec{q}$: unit vector along axis

θ : angle of rotation

$$q = \left[\cos(\theta/2), \sin(\theta/2)\vec{q}\right]$$

Unit quaternion of rotation

$$v = \left[0, \vec{v}\right]$$

$$qvq^c = \left[0, Q\vec{v}\right]$$

To effect rotation, convert vector to quaternion form and compute product, then convert back to vector form

By combining the previous expressions, we can find the rotation matrix Q associated with a quaternion q:

$$qvq^c = A_R(q^c)(A_L(q)v)$$

$$= \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

By multiplying these, and omitting the first row and column of the product (since we are only interested in what happens to pure vectors, which have no first component), we find:

$$qvq^c = A_R(q^c)(A_L(q)v) \rightarrow \vec{v}' = Q\vec{v} =$$

$$\begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Q is an ORTHOGONAL matrix, i.e. its transpose is its inverse:

$$QQ^T = Q^TQ = I$$

```
function animate(object)
% set object in prescribed motion
[s1,s2,s3]=sphere(10); axon = [0;1;1];angle=0;
[R,Index,X,Y,Z] = feval(object); % get object definition
% reorient object by prescribed rotation-translation
d0=[.1;.1;.1];v0=[.3,.3,1.4]; d=d0;   % define IC
orbit = []; kk=1;
while d(3) >= -5
   angle = kk * pi / 12; d = d0+kk*v0-.5*(kk/2)^2*[0;0;1];
   for i = 1:length(R)
      zz=d+rotate(R(i,:)',axon,angle);r(i,:)=zz';
   end
   x(1,:)=d';y(1,:)=d';z(1,:)=d';
   zz=d+rotate(X(2,:)',axon,angle);x(2,:)=zz';
   zz=d+rotate(Y(2,:)',axon,angle);y(2,:)=zz';
   zz=d+rotate(Z(2,:)',axon,angle);z(2,:)=zz';
   figure; hold on; grid on
   axis([-10,10,-10,10,-10,10]);  view(20,40);
```

```
    for =1:length(R)
         vcontour3(.1*s1+r(i,1),.1*s2+r(i,2),.1*s3+r(i,3)); end
    g=r(1,:);
    for i=2:length(Index)
       g=[g;r(Index(i),:)];  end
    plot3(g(:,1),g(:,2),g(:,3),'k-','LineWidth',1)
    plot3(x(:,1),x(:,2),x(:,3),'r-');
    plot3(y(:,1),y(:,2),y(:,3),'g-');
    plot3(z(:,1),z(:,2),z(:,3),'b-');
    plot3(orbit(:,1),orbit(:,2),orbit(:,3),'--om')
     hold off;pause(1);kk = kk+1;
end
function [R,Index,X,Y,Z]=octa1
s2=sqrt(2);
R=[1,1,0;1,-1,0;-1,-1,0;-1,1,0;0,0,s2;0,0,-s2];
X=[0,0,0;2,0,0];  Y=[0,0,0;0,2,0];  Z=[0,0,0;0,0,2];
Index=[1,2,3,4,1,5,3,6,1,2,5,4,6,2];
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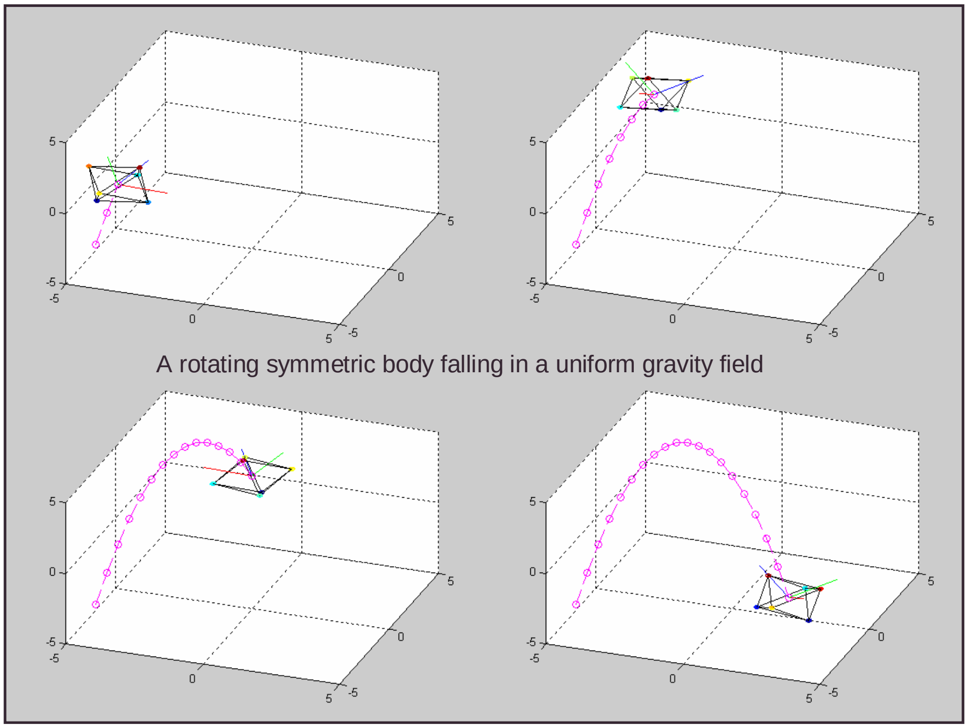

A rotating symmetric body falling in a uniform gravity field