

$$\underline{4.2.1} \rightarrow \{A^{4 \times 4}, \det A = 1/2\} \begin{cases} \text{(i) } \det(2A) = 2^4 \det A = 8 \\ \text{(ii) } \det(-A) = (-1)^4 \det A = 1/2 \\ \text{(iii) } \det A^2 = (\det A)^2 = 1/4 \\ \text{(iv) } \det(A^{-1}) = 1/\det A = 2 \end{cases}$$

$$\underline{4.2.21} \rightarrow \det A^{-1} = \det \left\{ \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right\} = (\text{correctly!}) \frac{1}{(ad-bc)^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{ad-bc}$$

$$\underline{4.2.29} \rightarrow P = A(A^T A)^{-1} A^T \Rightarrow |P| = |A(A^T A)^{-1} A^T|$$

To continue, need A square. In general $A^{n \times m}, n \neq m$.

$$\underline{4.2.30} \rightarrow f(a, b, c, d) = \ln(ad-bc)$$

$$\frac{\partial f}{\partial a} = \frac{d}{ad-bc}, \quad \frac{\partial f}{\partial b} = -\frac{c}{ad-bc}$$

$$\frac{\partial f}{\partial c} = \frac{-b}{ad-bc}, \quad \frac{\partial f}{\partial d} = \frac{a}{ad-bc}$$

$$\left(\frac{\partial f}{\partial a_{ij}} \right) = \frac{1}{\det A} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = A^{-1}$$

$\nearrow \frac{1}{\det A} A_{ij}$

In general: $\det A = a_{ij} C_{ij} + B = (-1)^{i+j} a_{ij} \det A_{ij} + B$

so that $\frac{\partial \det A}{\partial a_{ij}} = \frac{\partial \ln \det A}{\partial \det A} \frac{\partial \det A}{\partial a_{ij}} = \frac{1}{\det A} C_{ij}$

4.2.10) if $Q^T Q = I \Rightarrow \det Q^T \det Q = (\det Q)^2 = 1 \Rightarrow$
 $\det Q = \pm 1$; the parallelepiped is actually a
 unit cube.

4.2.11) $\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} = \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c^2-a^2) - (b-a)(c-a) \end{pmatrix}$

$= \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{pmatrix} = (a-b)(b-c)(c-a)$

$(c^2-a^2) - (b-a)(c-a)$
 $\parallel$
 $(c-a)(c-b)$

4.2.12) $CD = -DC$; $\det(CD) = \det(-DC)$
 $\Rightarrow \det C \det D = (-1)^n \det D \det C = 0$ if n odd.

$\uparrow$
 (flaw)

Set 9 Solutions

#4.3.3 (1) $\det(S^{-1}AS) = \det(S^{-1}) \det A \det S = (\det S)^{-1} (\det S) \det A = \det A$ True

(2) False; $\det A = 0 \Rightarrow$ columns of A dependent; this implies nothing about the

(3) False: $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 + 1 = 2.$

#4.3.6 $A_1 = (1)$, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$; find $D_n = \det A_n$.

(a) $D_n = \det \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & \dots & 1 \end{pmatrix} = D_{n-1} - \det \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & \dots & 0 \end{pmatrix} = D_{n-1} - D_{n-2}$

(b) So, $D_1 = 1$, $D_2 = 0$, $D_3 = -1$, $D_4 = -1$, $D_5 = 0$, $D_6 = 1$, $D_7 = 1$,

$D_8 = 0$. Since the D_n are generated by a two-term recursion, recurrence of two successive elements implies a cycle. Thus the D_n cycle with period 6, and $1000 = 166 \times 6 + 4$, so $D_{1000} = -1$.

#4.3.9 After some judiciously performed rearrangements we can see that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}$$

$$- \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} - \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

(this is actually the 4×4 case of a general theorem).

Clearly, if $C = \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, ~~det A = det B~~

Then $\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \det D$; as can be seen by from the expansion given above, $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det D + \det B \det C +$
 $+ (4 \text{ other terms involving the other compl. minors of order } 2).$

4.3.12 We saw that

$$\det A = \sum_{i_1, i_2, \dots, i_n=1}^n \epsilon_{i_1 i_2 \dots i_n} a_{i_1 1} a_{i_2 2} \dots a_{i_n n}$$

where the skew-symmetric tensor $\epsilon_{i_1 i_2 \dots i_n}$ is defined by $\epsilon_{i_1 i_2 \dots i_n} = \begin{cases} +1 & \text{even } \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \\ -1 & \text{odd } \end{cases}$
 $\bigcirc$ repeated indices

The sign associated to any given combination depends on whether it corresponds to an even or odd permutation, as follows:

$\epsilon_{54321} a_{15} a_{24} a_{33} a_{42} a_{51}$,
the desired term, corresponds to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \equiv (15)(24)$$

i.e. two exchanges, $1 \leftrightarrow 5$ and $2 \leftrightarrow 4$ can accomplish this permutation, which is therefore even and $\epsilon_{54321} = +1$.

In general, for an $n \times n$ matrix, the counterdiagonal has an even/odd permutation according to whether n is even/odd. $(-1)^{\lfloor \frac{n}{2} \rfloor} = \pm 1$ ($\lfloor \alpha \rfloor$ is the greatest integer $\leq \alpha$). — Here $(-1)^{\lfloor \frac{5}{2} \rfloor} = (-1)^2 = +1 \Rightarrow \text{even}$.