18.06 **Problem Set 10**

due: Wednesday, 9 May 2001

1. (10pts.) Determine the singular value decompositions of the following matrices (by hand calculation).

$$(a)\ \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix};\quad (b)\ \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix};\quad (c)\ \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};\quad (d)\ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix};\quad (e)\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

2. (10pts.) Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper (not exactly a standard mathematical transformation!). Do A and B have the same singular values? Prove that the answer is yes, or give a counterexample.

3. (10pts.)

   (a) Which of the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$ are linear transformations?

      i. $T(x, y, z) = (x + y - z, 2x + y)$;
      ii. $T(x, y, z) = (x + 1, y + z)$;
      iii. $T(x, y, z) = (0, \sqrt{3}z)$;
      iv. $T(x, y, z) = (x^2, y + z)$.

   (b) Let V be the vector space of 2×2-matrices over $\mathbb{R}$, and let B be a fixed matrix. Which of the following functions are linear transformations?

      i. $T(A) = AB$;
      ii. $T(A) = (\det A)B$;
      iii. $T(A) = A^2$;
      iv. $T(A) = BA$.

4. (10pts.) Let V be the vector space of all real polynomials of degree at most 4 and let $T : V \to V$ be the map $f(x) \mapsto 2f(x) - (2x + 2)f'(x) + x^2 f''(x)$.

   (a) Verify that T is a linear map.

   (b) Find the kernel and image of T and their dimensions.

   (c) Let B be the basis $1, x, x^2, x^3, x^4$ of V. Find the matrix $A = {}_B(T)_B$ of T with respect to this basis.

   (d) Find the rank of A and the dimension of the nullspace of A.

5. (10pts.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T : (x, y) \mapsto (x + 2y, 3x + 2y)$ and I be the identity map. Let E be the standard basis $\{(1, 0), (0, 1)\}$ and B be the basis $\{(1, 3), (2, 5)\}$. Let $\mathbf{v} = (1, 1)$.

   (a) Find the matrices $A = {}_E(T)_E$, which is the matrix of T in the basis E, and ${}_B(T)_B$, which is the matrix of T in the basis B.

(b) Calculate ${}_B(\mathbf{v})$, i.e. the vector $\mathbf{v}$ expressed in the basis B and ${}_B(T\mathbf{v})$, the vector $T\mathbf{v}$ expressed in the basis B.

(c) Find the change of basis matrices from basis E to basis B, ${}_B(I)_E$ and from basis B to basis E, ${}_E(I)_B$.

(d) Verify that ${}_B(T)_B = {}_B(I)_{E\,E}(T)_{E\,E}(I)_B$.

(e) Find a basis C such that $D = {}_C(T)_C$ is diagonal.

(f) Calculate $P = {}_E(I)_C$ and P^{-1}.

(g) Find a formula for $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}^n$ for any positive integer n.