18.06 **Hints and Answers to Problem Set 2**

1. (a) $\mathbf{h}^T = (-a, 4a, 7a)$ for any $a \in \mathbb{R}$.

   (b) $\mathbf{p}^T = (5/7, 1/7, 0)$ will do, and $\mathbf{h}$ is in part (a).

   (c) $\mathbf{c}^T = (1, 1, 0)$ or any $\mathbf{c}^T = (d, e, f)$ where $f \neq 3d - 2e$.

2. Let $C = A^T = [c_{ji}]$ and $D = B^T = [d_{kj}]$, where $c_{ji} = a_{ij}$ and $d_{kj} = b_{jk}$. Then the ki-entry of $B^T A^T$ is

$$\begin{aligned} d_{k1}c_{1i} + \cdots + d_{kn}c_{ni} &= b_{1k}a_{i1} + \cdots + b_{nk}a_{in} \\ &= a_{i1}b_{1k} + \cdots a_{in}b_{nk} \end{aligned}$$

   which is the ik-entry of AB.

3. (a) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. The $1j$-entry of AD is $2d_{1j} + 0d_{2j}$, and the $2j$-entry is $0d_{1j} + 1d_{2j}$.

   (b) $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The $1j$-entry of BD is $0d_{1j} + 1d_{2j}$, and the $2j$-entry is $1d_{1j} + 0d_{2j}$.

   (c) $C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The $1j$-entry of CD is $1d_{1j} + 0d_{2j}$, and the $2j$-entry is $1d_{1j} + 1d_{2j}$.

4. (a) $B = IB = (CA)B = C(AB) = CI = C$. Replacing B by A^{-1} we get $C = A^{-1}$. So the inverse is unique if it exists.

   (b) Let $A^{-1} = B$. Then $AB = BA = I$. Applying the definition of inverse to B, we see that $A = B^{-1}$.

   (c) $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$. Similarly, $ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$.

   (d) Given $A\mathbf{x} = \mathbf{b}$, multiply by A^{-1} to get $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$. Now, $A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x}$. So $\mathbf{x} = A^{-1}\mathbf{b}$ is uniquely determined.