18.06 **Hints and Answers to Problem Set 9**

1.

$$\begin{aligned}
(f, A[g]) &= \int_0^1 f^* A[g]\, dx = \int_0^1 f^* \left(\frac{d}{dx}\left(p\frac{dg}{dx}\right) + qg \right) dx \\
&= \left[f^* p \frac{dg}{dx} \right]_0^1 - \int_0^1 \frac{df^*}{dx} p \frac{dg}{dx}\, dx + \int_0^1 f^* qg\, dx \\
&= -\left[\frac{df^*}{dx} pg \right]_0^1 + \int_0^1 \frac{d}{dx}\left(\frac{df^*}{dx} p \right) g\, dx + \int_0^1 f^* qg\, dx \\
&= \int_0^1 \left(\frac{d}{dx}\left(\frac{df^*}{dx} p \right) + qf^* \right) g\, dx \\
&= (A[f], g)
\end{aligned}$$

The boundary terms vanish, due to the boundary conditions.

Let A be hermitian, and $A[f] = \lambda f$. Then

$$\begin{aligned}
\lambda(f, f) &= (f, \lambda f) = (f, A[f]) \\
&= (A[f], f) = (\lambda f, f) \\
&= \lambda^*(f, f)
\end{aligned}$$

By definition of an eigenfunction (eigenvector), $(f, f) \neq 0$, and so $\lambda = \lambda^*$.

Let A be hermitian, and $A[f_1] = \lambda_1 f_1$, $A[f_2] = \lambda_2 f_2$ and $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned}
\lambda_1(f_1, f_2) &= (\lambda_1 f_1, f_2) = (A[f_1], f_2) \\
&= (f_1, A[f_2]) = (f_1, \lambda_2 f_2) \\
&= \lambda_2(f_1, f_2)
\end{aligned}$$

So we find that $(\lambda_1 - \lambda_2)(f_1, f_2) = 0$ and since $\lambda_1 \neq \lambda_2$, $(f_1, f_2) = 0$.

2. Let $A = MBM^{-1}$.

   (a) $\mathrm{Tr}A = \mathrm{Tr}[MBM^{-1}] = \mathrm{Tr}[(MB)M^{-1}] = \mathrm{Tr}[M^{-1}(MB)] = \mathrm{Tr}[(M^{-1}M)B] = \mathrm{Tr}[IB] = \mathrm{Tr}B$.

   (b)

$$\begin{aligned}
\det A &= \det(MBM^{-1}) = (\det M)(\det B)(\det M^{-1}) \\
&= (\det M \det M^{-1}) \det B = \det(MM^{-1}) \det B \\
&= \det I \det B = 1 \cdot \det B \\
&= \det B.
\end{aligned}$$

   (c) Suppose that $B\mathbf{x} = \lambda\mathbf{x}$. Then $A(M\mathbf{x}) = MBM^{-1}(M\mathbf{x}) = MB\mathbf{x} = M(\lambda\mathbf{x}) = \lambda(M\mathbf{x})$. So $M\mathbf{x}$ is an eigenvector of A with the same eigenvalue λ. Similarly, suppose that $A\mathbf{y} = \eta\mathbf{y}$. By an analogous argument it follows that $M^{-1}\mathbf{y}$ is an eigenvector of B with the same eigenvalue η.

3. Notice that $\mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(AB) = 0$ while on the other hand, $\mathrm{Tr}(I_n) = n \neq 0$.