18.06 **Problem Set 9**

due: Wednesday, 2 May 2001

1. (10pts.) An important (infinite-dimensional) vector space, call it $\mathcal{H}$, is the space of all complex-valued functions $f(x)$, defined on the interval $[0,1]$ such that $\int_0^1 |f|^2 dx < \infty$. An inner product in $\mathcal{H}$ is defined by

$$(f,g) = \int_0^1 f^* g\, dx.$$

We are interested in second order linear differential operators acting on $\mathcal{H}$. A general such operator takes the form:

$$L[f] \equiv a_0(x)f'' + a_1(x)f' + a_2(x)f.$$

An operator A acting on $\mathcal{H}$ is called hermitean, if

$$(f, A[g]) = (A[f], g) \quad \text{for all } f, g \in \mathcal{H}.$$

Show that operators of the form

$$A[f] \equiv \frac{d}{dx}\left(p(x)\frac{df}{dx}\right) + q(x)f$$

where $p(x)$ and $q(x)$ are real-valued functions, and with boundary conditions $p(0) = p(1) = 0$ are hermitean. (*Hint:* You will have to use integration by parts.)

Now consider the eigenvalue problem

$$A[f] = \lambda f \quad \text{for } 0 < x < 1.$$

Show that, if A is hermitean, eigenvalues are real and that eigenfunctions corresponding to different eigenvalues are orthogonal, i.e. for two such functions $f_1(x), f_2(x)$, we have $(f_1, f_2) = 0$.

2. (10pts.) Show that any two similar matrices have the same
   (a) trace;
   (b) determinant;
   (c) eigenvalues.

3. (10pts.) Show that there are no two $(n \times n)$ matrices A, B such that $AB - BA = I_n$.