18.06 **Hints and Answers to Problem Set 1**

1. (a) $ax + by + cz = d$ for constants $a, b, c, d \in \mathbb{R}$.
   (b) $\mathbf{n} = \nabla(ax + by + cz) = (a, b, c)$.
   (c) For any vector $\mathbf{x} = (x, y, z)$ pointing to the plane, $\mathbf{n} \cdot (\mathbf{x} - (1, 0, 1)) = 0$. So
$$2(x-1) - 1(y-0) + 3(z-1) = 0 \iff 2x - y + 3z = 5$$
   (d) $\mathbf{n} = \vec{PQ} \times \vec{PR} = (1, 2, 2) \times (1, 0, -1) = (-2, 3, -2)$ Now proceed as in part c) to find $-2x + 3y - 2z = -3$.
   (e) The shortest distance from the origin to the plane is along the direction of the normal vector to the plane.The line through the origin along $(-2, 3, -2)$ is $\{t(-2, 3, -2) | t \in \mathbb{R}\}$ and this line intersects the plane at the following point:
$$-2(-2t) + 3(3t) - 2(-2t) = -3 \iff t = -\frac{3}{17}$$
   The length of the vector $-(3/17)(-2, 3, -2)$ is $(3/17)\sqrt{4+9+4} = 3/\sqrt{17}$.

2. (a) a point (generic case)
   (b) a line
   (c) a plane (three coinciding planes)
   (d) no intersection ("toblerone" tube of three planes intersecting in three lines)
   (e) no intersection (parallel planes)

3. (a) Simply multiply out.
   (b)
$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & az \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
   (c)
$$\gamma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \gamma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

4. (a) General solution is $(4, 0, 3, -1) + (2a, a, 0, 0)$ (any $a \in \mathbb{R}$).
   (b) No solutions – echelon form: $\left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right)$
   (c) General solution: $(-8, -23, -7, 0) + (0, -5a, a, 2a)$ (any $a \in \mathbb{R}$). For $x_2 = -3$, take $a = -4$ to get $(-8, -3, -11, -8)$.
   (d) General solution is $(5a, -2a, a)$.

5. Echelon form is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & a-2 & 3 \\ 0 & 0 & -2(a-1) & b-3 \end{array}\right)$

(a) No solutions when $a = 1$ and $b \neq 3$.
(b) Exactly one solution when $a \neq 1$.
(c) Finitely many, but at least two solutions: never.
(d) Infinitely many solutions when $a = 1$ and $b = 3$.