18.06 Hints and Answers to Problem Set 10

1. (a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (c) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

   (e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

2. Write

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} a_{m1} & \cdots & a_{21} & a_{11} \\ a_{m2} & \cdots & a_{22} & a_{12} \\ a_{m3} & \cdots & a_{23} & a_{13} \\ \vdots & \ddots & \vdots & \vdots \\ a_{mn} & \cdots & a_{2n} & a_{1n} \end{pmatrix}.$$

If we denote the element of B in its ith row and jth column by b_{ij}, then we have $b_{ij} = a_{(m-j+1),i}$. Now, the singular values of A are the positive square roots of the non-zero eigenvalues of A^TA, or AA^T, and similarly for B. Notice that the ij element of BB^T is

$$(BB^T)_{ij} = \sum_{k=1}^{m} b_{ik}b_{jk} = \sum_{k=1}^{m} a_{(m-k+1),i}a_{(m-k+1),j} = \sum_{\ell=1}^{m} a_{\ell i}a_{\ell j} = (A^TA)_{ij}.$$

Here we have changed the dummy summation variable, $\ell = m - k + 1$, and found that $BB^T = A^TA$. So these two matrices are the same, hence they have the same eigenvalues, and it follows that A and B have the same singular values.

3. (a)
   i. linear;
   ii. not linear;
   iii. linear;
   iv. not linear.

   (b)
   i. linear;
   ii. not linear;
   iii. not linear;
   iv. linear.

4. (a)

$$\begin{aligned} T(f+g) &= 2(f+g) - (2x+2)(f+g)' + x^2(f+g)'' \\ &= 2f - (2x+2)f' + x^2f'' + 2g - (2x+2)g' + x^2g'' \\ &= T(f) + T(g) \end{aligned}$$

$$\begin{aligned} T(cf) &= 2(cf) - (2x+2)(cf)' + x^2(cf)'' \\ &= c(2f - (2x+2)f' + x^2 f'') \\ &= cT(f) \end{aligned}$$

(b) $T(a+bx+cx^2+dx^3+ex^4) = 2(a-b) - 4cx - 6dx^2 + 2(d-4e)x^3 + 6ex^4$ so we see that Ker$T = \text{Sp}\{1+x\}$; dim Ker$T$=1. Also, Im $T = \text{Sp}\{1, x, -6x^2+2x^3, -8x^3+6x^4\}$; dim Im$T$=4.

(c)

$$A = {}_B(T)_B = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}.$$

(d) Null Space=Sp$\{(1,1,0,0,0)^T\}$, so nullity=1. There are 4 pivot columns, so the rank is 4. And $4+1=5$, the dimension of V as expected.

5. (a)

$${}_E(T)_E = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$T(1,3) = (7,9) = (-17)(1,3) + 12(2,5)$
$T(2,5) = (12,16) = (-28)(1,3) + 20(2,5)$ Thus,

$${}_B(T)_B = \begin{pmatrix} -17 & -28 \\ 12 & 20 \end{pmatrix}.$$

(b)

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (-3)\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 5 \end{pmatrix} \qquad \text{So} \quad {}_B(\mathbf{v}) = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$${}_B(T\mathbf{v}) = {}_B(T)_{BB}(\mathbf{v}) = \begin{pmatrix} -17 & -28 \\ 12 & 20 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$$

(c)

$${}_E(I)_B = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}; \qquad {}_B(I)_E = ({}_E(I)_B)^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

(d)

$${}_B(T)_B = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -17 & -28 \\ 12 & 20 \end{pmatrix}$$

(e)

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = 0 \iff \lambda_{1,2} = 4, -1;$$

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(f)

$$P = {}_E(J)_C = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \qquad P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$$

(g)

$$A^n = PD^nP^{-1} = \frac{1}{5}\begin{pmatrix} 2\cdot 4^n + 3\cdot(-1)^n & 2\cdot 4^n - 2\cdot(-1)^n \\ 3\cdot 4^n - 3\cdot(-1)^n & 3\cdot 4^n + 2\cdot(-1)^n \end{pmatrix}$$