18.06 **Hints and Answers to Problem Set 8**

1. $\mathrm{Tr}(A) = (3/5) - (3/5) = 0$, $\det A = -(9/25) - (16/25) = -1$. The inverse of A is just A itself, $A^{-1} = A$. A matrix, which is its own inverse is called *idempotent*. Eigenvalues and eigenvectors are

$$\lambda_1 = 1,\ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \lambda_2 = -1,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

2. (a) The trace is, $\mathrm{Tr}(A) = -3 - 6 - 3 = -12$. The determinant is, $\det A = 98$.

   (b)

$$A^{-1} = \frac{1}{98}\begin{pmatrix} 14 & 14 & 28 \\ 14 & -7 & 14 \\ 28 & 14 & 14 \end{pmatrix}$$

   (c) The characteristic equation is $\lambda^3 + 12\lambda^2 + 21\lambda - 98 = 0$. Factoring out $(\lambda - 2)$ from the left hand side, we are left with $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2 = 0$. So there is a double root, $\lambda = -7$. There is one eigenvector corresponding to $\lambda_1 = 2$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, and there is a two-dimensional eigenspace corresponding to $\lambda_{2,3} = -7$, which is given by $2x + y + 2z = 0$. A vector in this space is for example, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$.

3. First, we need to find the eigenvalues, which we do by solving the characteristic equation:

$$\det(A - \lambda I) = \lambda^4 - 4\lambda^3 - 48\lambda^2 + 320\lambda - 512 = 0$$

From the determinant, it should be obvious that it is singular if $1 - \lambda = -3 \iff \lambda = 4$. In fact this is a triple root of the characteristic equation, which has roots $\lambda_{1,2,3} = 4$ and $\lambda_4 = -8$. Corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Eigenvectors corresponding to different eigenvalues are necessarily orthogonal, but those corresponding to the eigenvalue 3 are not yet orthogonal. We have to use Gram-Schmidt to make them orthogonal, and we find that

$$\tilde{\mathbf{v}}_2 = \sqrt{\frac{2}{3}}\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \tilde{\mathbf{v}}_3 = \frac{\sqrt{3}}{2}\begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ \frac{1}{3} \end{pmatrix}.$$

Then the vectors $\mathbf{v}_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3$ and $\mathbf{v}_4$ make up the columns of the matrix P.

4. (a) True: Suppose $A\mathbf{x} = \lambda\mathbf{x}$. Then $(A - \mu I)\mathbf{x} = A\mathbf{x} - \mu I\mathbf{x} = \lambda\mathbf{x} - \mu\mathbf{x} = (\lambda - \mu)\mathbf{x}$.

   (b) False: For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ has eigenvalues 1 and 2, but not -1 and -2.

   (c) True: We can write $A = SJS^{-1}$ where J is a matrix in Jordan form, with the (nonzero) eigenvalues of A on its diagonal. Then $A^{-1} = SJ^{-1}S^{-1}$. Now,

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix} \Rightarrow J^{-1} = \begin{pmatrix} J_1^{-1} & & & \\ & J_2^{-1} & & \\ & & \ddots & \\ & & & J_r^{-1} \end{pmatrix}$$

   For the $p \times p$ Jordan block J_i it is easy to see that

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix} \Rightarrow J_i^{-1} = \begin{pmatrix} \frac{1}{\lambda_i} & -\frac{1}{\lambda_i^2} & \cdots & \frac{(-1)^{p+1}}{\lambda_i^p} \\ & \frac{1}{\lambda_i} & \cdots & \frac{(-1)^p}{\lambda_i^{p-1}} \\ & & \ddots & \vdots \\ & & & \frac{1}{\lambda_i} \end{pmatrix}$$

   So J^{-1} is also an upper triangular matrix, which therefore has its eigenvalues on the diagonal. These eigenvalues are just the reciprocals of the eigenvalues of A. Since A^{-1} is similar to J^{-1}, they are also eigenvalues of A^{-1}.

   (d) False: For example, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is a nonzero matrix, which has all eigenvalues equal to zero.

   (e) True: We can write $A = S\Lambda S^{-1}$ and since all the eigenvalues are equal, $\Lambda = \lambda I$. Then $A = S(\lambda I)S^{-1} = \lambda SIS^{-1} = \lambda SS^{-1} = \lambda I$, which is diagonal.