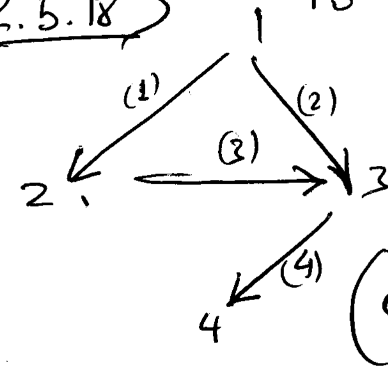
2.5.18 To demonstrate ideas, consider graph



$$A = \begin{array}{c} \text{nodes} \longrightarrow \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \\ \downarrow \text{edges} \end{array}, \quad A^T = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Note that $(A^TA)_{ii} = \langle a_{i*}, a_{i*} \rangle = \sum_{k=1}^{n} a_{ik}^2$. ($\dagger$)

In general, A is $m \times n$ (m = # edges, n = # nodes)
Here $m = n = 4$

For A above:
$(A^TA)_{11} = 2$, $(A^TA)_{22} = 2$
$(A^TA)_{33} = 3$, $(A^TA)_{44} = 1$

Since the entries of column k of A are 0 unless there is an edge entering or leaving the node k, (in which case a_{ik}, $i = 1, .., m$ is ± 1 etc) we have that the above sum ($\dagger$) = number of edges entering or leaving that node.

(b) $$A^TA = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$ we see (-1) occurs when two nodes are connected and (0) when they are not

In general (the i-th row of A^T = i-th col. of A)

$$\langle a_{*i}, a_{*k} \rangle = \sum_{l=1}^{m} a_{li} a_{lk} = (A^TA)_{ik}$$

Now, the i-th column has (± 1) where edges enter or leave node i, same for column k and node k.
Then the only time these two will have a non zero product is for l = # of edge connecting these two nodes. If no connection, $(A^TA)_{ik} = 0$.
If there is a connection, then one of a_{li}, a_{lk} is entry $(+1)$ the other is (-1) (exit) ; then $(A^TA)_{ik} = (1)(-1) = -1$.