18. In the vector space P_3 of all $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, let **S** be the subset of polynomials with $\int_0^1 p(x)\,dx = 0$. Verify that **S** is a subspace and find a basis.

$$\int_0^1 p(t)\,dt = 0 \Rightarrow a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \frac{a_3 t^4}{4}\Big|_0^1$$

$$= a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 = 0$$

i.e. $$\left(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}\right)\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

i.e. P_3 is the null space of the rank 1 matrix

$$A = [1 \;\; \tfrac{1}{2} \;\; \tfrac{1}{3} \;\; \tfrac{1}{4}]$$

Consider $A^T = \begin{array}{r} \\ -\frac{1}{2} \\ -\frac{1}{3} \\ -\frac{1}{4} \end{array}\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$ $\cancel{=}$ to perform LU factorization: *

$$\begin{pmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ -\frac{1}{3} & 0 & 1 & \\ -\frac{1}{4} & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e. left null vectors of A^T are

$$N(A) = \underset{\text{span}}{\left\{\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}}$$

(* i.e. null vectors of A)

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47. The 4 by 4 *Hadamard matrix* is entirely +1 and −1:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Find H^{-1} and write $v = (7, 5, 3, 1)$ as a combination of the columns of H.

Since $HH = I$, we have that

$$H\vec{x} = v \Rightarrow x = Hv = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \\ 8 \\ 0 \end{pmatrix}$$

i.e. $\boxed{v = 7 \cdot H_1 + 5 H_2 + 3 H_3}^{4}$