4.2.1 $\left\{A^{4\times 4},\ \det A = 1/2\right\}$ (i) $\det(2A) = 2^4 \det A = 8$

(ii) $\det(-A) = (-1)^4 \det A = 1/2$

(iii) $\det A^2 = (\det A)^2 = 1/4$ (iv) $\det(A^{-1}) = 1/\det A = 2$

4.2.21 $\det A^{-1} = \det\left\{ \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right\} = (\text{correctly!})\ \frac{1}{(ad-bc)^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{ad-bc}$

4.2.29 $P = A(A^TA)^{-1}A^T \Rightarrow |P| = |A(A^TA)^{-1}A^T|$

To continue, need A square. In general $A^{n\times m}$, $n\neq m$.

4.2.30 $f(a,b,c,d) = \ln(ad-bc)$

$\frac{\partial f}{\partial a} = \frac{d}{ad-bc}$, $\frac{\partial f}{\partial b} = -\frac{c}{ad-bc}$

$\frac{\partial f}{\partial c} = \frac{-b}{ad-bc}$ $\frac{\partial f}{\partial d} = \frac{a}{ad-bc}$

$\frac{1}{\det} A_{ij}$

$$\left(\frac{\partial f}{\partial a_{ij}}\right) = \frac{1}{\det A}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = A^{-1}$$

In general: $\det A = a_{ij}C_{ij} + B = (-1)^{i+j} a_{ij} \det A_{ij} + B$

so that $\frac{\partial \ln \det A}{\partial a_{ij}} = \frac{\partial \ln \det A}{\partial \det A}\frac{\partial \det A}{\partial a_{ij}} = \frac{1}{\det A} C_{ij}$

$\cancel{4.2.9}$/10) if $Q^TQ = I \Rightarrow \det Q^T \det Q = (\det Q)^2 = 1 \Rightarrow$
$\det Q = \pm 1$; the parallel~~ogram~~epiped is actually a unit cube.

$\cancel{4.2.10}$/12) $\det\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = \det\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} = \det\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & \otimes \end{pmatrix}$

$\otimes = (c^2-a^2) - (b+a)(c-a) = (c-a)(c-b)$

$= \det\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{pmatrix} = (a-b)(b-c)(c-a)$

$\cancel{4.2.17}$/19) $CD = -DC$; $\det(CD) = \det(-DC)$
$\Rightarrow \det C \det D = (-1)^n \det D \det C = 0$ if n odd.

(flaw)

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H W # 7.

89/90 L1

Ex. 4.2.(3/4) By applying row operations to produce an upper triangular U, compute:

$$\det\begin{pmatrix}1&2&-2&0\\2&3&-4&1\\-1&-2&0&2\\0&2&5&3\end{pmatrix}=\det\begin{pmatrix}1&2&-2&0\\0&-1&0&1\\0&0&-2&2\\0&2&5&3\end{pmatrix}=$$

$$=\det\begin{pmatrix}1&2&-2&0\\0&-1&0&1\\0&0&-2&2\\0&0&5&5\end{pmatrix}=\det\begin{pmatrix}1&2&-2&0\\0&-1&0&1\\0&0&-2&2\\0&0&0&10\end{pmatrix}=$$

$$=1\cdot(-1)(-2)\cdot 10=20$$

$$\det\begin{pmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{pmatrix}=-\det\begin{pmatrix}-1&2&-1&0\\2&-1&0&0\\0&-1&2&-1\\0&0&-1&2\end{pmatrix}=$$

$$=-\det\begin{pmatrix}-1&2&-1&0\\0&3&-2&0\\0&-1&2&-1\\0&0&-1&2\end{pmatrix}=+\det\begin{pmatrix}-1&2&-1&0\\0&-1&2&-1\\0&3&-2&0\\0&0&-1&2\end{pmatrix}=$$

$$=+\det\begin{pmatrix}-1&2&-1&0\\0&-1&2&-1\\0&0&4&-3\\0&0&-1&2\end{pmatrix}=-\det\begin{pmatrix}-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\\0&0&4&-3\end{pmatrix}=$$

$$=-\det\begin{pmatrix}-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\\0&0&0&5\end{pmatrix}=-(-1)(-1)(-1)\cdot 5=5$$

✓

$$\det\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & 2 & -1 \end{pmatrix} = -\det\begin{pmatrix} -1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & 2 & -1 \end{pmatrix} =$$

$$= -\det\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & 2 & -1 \end{pmatrix} = \det\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 3 & -2 & 0 \end{pmatrix} =$$

$$= \det\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 4 & -3 \end{pmatrix} = \det\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 5 \end{pmatrix} =$$

$$= (-1)(-1)(-1)\cdot 5 = -5$$

✓

Ex. 4.2 (10)(12) Use row operations to verify that the 3×3 "Vandermonde determinant" is

$$\underbrace{\det\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}}_{= V \text{ (define)}} = (b-a)(c-a)(c-b)$$

$$\det V = \det\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} = \left(\text{multiply the second row by } \frac{c-a}{b-a} \text{ and } (\mathrm{III} - \mathrm{II})\right)$$

$$= \det\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & \underbrace{\cancel{c^2}-\cancel{a^2}-bc-ac+ab+\cancel{a^2}}_{\substack{\| \\ c(c-a)-b(c-a)= \\ =(c-a)(c-b)}} \end{pmatrix} = \det\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

$$= \underline{(b-a)(c-a)(c-b)}$$

✓

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HW # 7.

Ex. 4.2. (11/13)

(a) A skew-symmetric matrix satisfies $K^T = -K$. In the 3×3 case why is $\det(-K) = (-1)^3 \det K$? On the other hand $\det K^T = \det K$ (always). Deduce that $-\det K = \det K$ and det must be zero.

$$\det K \overset{\text{always}}{=} \det K^T \overset{\text{for skew-symmetric}}{=} \det(-K) \qquad (*)$$

Check this for 3×3 case:

$$\det(-K) = \det\begin{pmatrix} (-1)\cdot \text{row } 1 \\ (-1)\cdot \text{row } 2 \\ (-1)\cdot \text{row } 3 \end{pmatrix} \underset{\substack{\text{apply property 1 of} \\ \text{determinants 3 times} \\ \text{with } t=-1.}}{=} (-1)^3 \det\begin{pmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{pmatrix} = (-1)^3 \det K$$

$\Rightarrow \det(-K) = -\det K$ for 3×3 case
from this and (*) $\Rightarrow \det K = -\det K$.
It means that $\det K = 0$. ✓

(b) Write down a 4×4 skew-symmetric matrix with $\det K \neq 0$.

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\det K = 1\cdot(-1)^{1+2} \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = -(-(-1)^2) = 1 \neq 0$$ ✓

Set 9 Solutions

#4.3.3 (1) $\det(S^{-1}AS) = \det(S^{-1})\det A \det S = (\det S)^{-1}(\det S)\det A = \det A$ True

(2) False; $\det A = 0 \Rightarrow$ columns of A dependent; this implies nothing about the rows of the cofactors.

(3) False: $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1+1 = 2.$

#4.3.6 $A_1 = (1)$, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$; find $D_n = \det A_n$.

(a) $$D_n = \det\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ & & \cdots & & & \end{pmatrix} = D_{n-1} - \det\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 \cdots & 0 \\ 0 & 1 & 1 & 1 \cdots & 0 \\ \cdot & \cdots & & & \cdot \end{pmatrix} = D_{n-1} - D_{n-2}$$

← has cofactor with column of zeroes

(b) So, $D_1 = 1$, $D_2 = 0$, $D_3 = -1$, $D_4 = -1$, $D_5 = 0$, $D_6 = 1$, $D_7 = 1$, $D_8 = 0$; Since the D_n are generated by a two-term recursion, recurence of two successive elements implies a cycle. Thus the D_k cycle with period 6, ~~and~~ And $1000 = 166 * 6 + 4$, so $D_{1000} = -1$.

#4.3.~~9b~~(31) After some judiciously performed rearrangements we can see that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}$$

$$- \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} - \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

(this is actually the 4x4 case of a general theorem).

Clearly, if $C \equiv \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, ~~det A = det B d~~

Then $\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \det D$; as can be seen ~~by~~ from the expansion given above, $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det D - \det B \det C +$ + (4 other terms involving the other compl. minors of order 2).

4.3.~~12~~ We saw that

$$\det A = \sum_{i_1 \neq i_2 \ldots i_n = 1}^{n} \varepsilon_{i_1 i_2 \ldots i_n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$$

where the ~~con~~ skew-symmetric tensor $\varepsilon_{i_1 i_2 \ldots i_n}$ is defined by

$$\varepsilon_{i_1 i_2 \ldots i_n} = \begin{cases} +1 & \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & n \end{pmatrix} \text{ even} \\ -1 & \text{odd} \\ 0 & \text{repeated indices} \end{cases}$$

The sign associated to any given combination depends on whether it corresponds to an even or odd permutation, as follows:

$$\varepsilon_{54321}\, a_{15} a_{24} a_{33} a_{42} a_{51},$$

the desired term, corresponds to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \equiv (15)(24)$$

i.e. two exchanges, $1 \leftrightarrow 5$ and $2 \leftrightarrow 4$ can accomplish this permutation, which is therefore even and $\varepsilon_{54321} = +1$.

In general, for an $n \times n$ matrix, the counterdiagonal has an ~~od~~ even/odd permutation according to whether ~~n e is odd/even~~. $(-1)^{[\frac{n}{2}]} = \pm 1$ ($[\alpha]$ is the greatest integer $\leq \alpha$). — Here $(-1)^{[\frac{5}{2}]} = (-1)^2 = +1 \Rightarrow$ even.