Math 464/514
Set4

p. 110, Sec. 2.4
(1, 3, 5, 6, 14, 16, 17, 18, 27, 36)

2.4.14 Find left/right inverses (if they exist) for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$A = M^T$ so work with A: rank $A = 2 = m < n = 3$

A has full row rank, but not full column rank

$\Rightarrow \exists$ right inverse (non-unique), no left inverse

~~Crossed~~ Then use Gauss-Jordan

$$A A_R^{-1} = I^{2\times 2} \Rightarrow \left(\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

i.e. $\begin{pmatrix} x \\ y \end{pmatrix}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\begin{pmatrix} x \\ y \end{pmatrix}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

and $A_R^{-1} \left(= M_L^{-1,T}\right) = \begin{pmatrix} 1+t & -1+s \\ -t & 1-s \\ t & s \end{pmatrix}$

T is nonsingular (upper triangular) if $a \neq 0$; then $T_R^{-1} = T_L^{-1} = \dfrac{1}{a^2}\begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$ — No inverse otherwise ($r=1$)

2.4.16 Suppose A has a right inverse, B. Then $AB = I$ leads to $A^TAB = A^T$ or $B = (A^TA)^{-1}A^T$. But that satisfies $BA = I$, i.e. B is also a left inverse. What step is bogus?

(A^TA) may be singular

2.4.18 Find a basis for each of the four subspaces of

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$m = 3$, $n = 5$
$r = 2$ (pivots)

$\mathcal{R}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}\right\}$ (pivotal columns) dim $= 2 = r$

$\mathcal{R}(A^T) = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}\right\}$ (nonzero U-rows) dim $=$

dim $\mathcal{N}(A^T) = 1$

$$L^{-1} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \boxed{1 \;\; -1 \;\; 1} \end{pmatrix} \Rightarrow \mathcal{N}(A^T) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$$L = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow$$

$\mathcal{N}(A)$: ~~$Ux = 0$~~ $U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-3} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \end{array} \begin{pmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}x + \begin{pmatrix} -2 \\ 0 \end{pmatrix}z + \begin{pmatrix} 2 \\ -2 \end{pmatrix}v$$

$$\mathcal{N}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right\}$$

$x = 1$, $z = v = 0$ | $x = v = 0$, $z = 1$ | $x = z = 0$, $v = 1$

2.4.27 A is an $m\times n$ matrix, rank r. Suppose $Ax=b$ is not solvable for some b

(a) What inequalities ($<$ or $\leq$) must be true between m, n, r?

(b) How do you know that $A^T y = 0$ has a nonzero solution?

(a) If $\exists\, b \in \mathbb{C}^m$ with $b \notin \mathcal{R}(A) \Rightarrow r < m$ (no info. regarding n, except that $r \leq n$ which is always true).

(b) Since $\dim \mathcal{N}(A^T) + \underbrace{\dim \mathcal{R}(A)}_{=r} = m \Rightarrow \dim \mathcal{N}(A^T) = m - r > 0$

2.4.36 Without multiplying matrices, find bases for $\mathcal{R}(A)$, $\mathcal{R}(A^T)$, if $A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix} = LR$

How do you know from these shapes (= dimensions) that A is not invertible?

Now both L (3×2) and R (2×3) have full rank, $r=2$.

So, if $x \in \mathcal{N}(A) \Rightarrow Ax = 0 \Rightarrow L(Rx)=0$. Either $Rx = 0 \Rightarrow x \in \mathcal{N}(R)$ or $y = Rx$ satisfies $Ay = 0$. ~~A has to dim N(L)~~

The second possibility does not obtain, since $\dim \mathcal{N}(L) = n - r = 0 \Rightarrow \mathcal{N}(A) = \mathcal{N}(R)$ (subspace of same dimension). Similarly, $\mathcal{N}(A^T) = \mathcal{N}(L^T)$.

$\therefore \dim \mathcal{N}(A) = \dim \mathcal{N}(R) = n - r = 1 \Rightarrow A$ not invertible

Since $y \in \mathcal{R}(A) \Leftrightarrow \exists\, x:\ Ax = y$ we have

$Ax = y \Rightarrow L(Rx) = y \Rightarrow Lz = y$ with $z = Rx$

$\Rightarrow \{y \in \mathcal{R}(A) \Rightarrow y \in \mathcal{R}(L)\} \Rightarrow \mathcal{R}(A) \subset \mathcal{R}(L)$

If $y \in \mathcal{R}(L) \Rightarrow \exists\, z:\ Lz = y$. To show $y \in \mathcal{R}(A)$ there must exist $x:\ Rx = z$. This is guaranteed since R has full column rank $(r=m=2)$. So $y \in \mathcal{R}(L) \Rightarrow y \in \mathcal{R}(A)$ i.e. $\mathcal{R}(A) = \mathcal{R}(L)$.

$\therefore$ The columns of L give $\mathcal{R}(A)$, the rows of R give $\mathcal{R}(A^T)$.

$\Rightarrow A$ cannot have full rank if decomposed into rectangulars

2.4.1. $m = n \Rightarrow A$ square; but now a column space are subspaces of $\mathbb{R}^m$ with the same dimension, but not necessarily the same unless $A = A^T$.

2.4.3

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}} U = L_1 A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R(U) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\} ; \quad R(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$R(U^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\} = R(A^T)$$

$$N(U) = N(A): \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} x_2 = -\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x_3 - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_4$$

free vars: x_3, x_4

basic vars: x_1, x_2

$$\left.\begin{matrix} x_3 = 1 \\ x_4 = 0 \end{matrix}\right\} \quad \begin{cases} x_1 + 2x_2 = 0 \Rightarrow x_1 = 2 \\ x_2 = -1 \end{cases} \qquad u_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} x_3 = 0 \\ x_4 = 1 \end{matrix}\right\} \quad \begin{matrix} x_1 + 2x_2 = -1 \Rightarrow x_1 = -1 \\ x_2 = 0 \end{matrix} \qquad u_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$N(U) = N(A) = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{L_1 = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 0 & 0 & 1 & \\ -1 & 0 & 0 & 1 \end{pmatrix}} A_2^T = L_1 A^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & 0 & 0 & 1 \end{pmatrix}}$$

$$\rightarrow U = L_2 L_1 A^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad \begin{cases} y_1 + 0 = -1 \\ y_2 = 0 \end{cases} : \quad u_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$N(U^T) = \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ \ldots \end{pmatrix} \right\}$

$$U^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{L_1 = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 0 & 0 & 1 & \\ -1 & 0 & 0 & 1 \end{pmatrix}} L_1 U^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

basic: y_1, y_2; free y_3: $n_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \cancel{0} \end{pmatrix}$

$$N(U^T) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ \cancel{0} \end{pmatrix} \right\}$$

.4.5) $AB = 0 \Rightarrow A(b_1\, b_2 \cdots b_n) = 0 \Rightarrow (Ab_1, \dots, Ab_n) = 0$

$\Rightarrow Ab_i = 0,\ i = 1, \dots, n$; b_i : ith column of B

$\Rightarrow b_i \in N(A),\ i = 1, \dots, n \Rightarrow \text{span}\{b_i\} = R(B) \subset N(A)$.

Since $AB = (B^T A^T)^T = 0 \Rightarrow B^T A^T = 0$

similarly then $R(A^T) \subset N(B^T)$.

2.4.6) If $\text{rank}\, A = \text{rank}\, A' = \text{rank}\,(A, b)$

$\Rightarrow b$ is linearly dep. on columns of $A \Rightarrow b \in R(A)$.

$\Rightarrow Ax = b$ is solvable.

2.4.17) $AB = I \Rightarrow A^T AB = A^T$ ~~five~~

But $A^T A$ may be singular, so we cannot necessarily

write $B = (A^T A)^{-1} A^T$. ($A^T A$ will be certainly

2.4.19

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$$

Now, $\left.\begin{matrix} x_2, x_4 \text{ basic} \\ x_1, x_3, x_5 \text{ free} \end{matrix}\right\}$:

& $R(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}$; $R(A^T) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$

$N(A)$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} x_4 = -\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} x_1 - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} x_3 - \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} x_5$

n_1: $x_2 = x_4 = 0$; $x_1 = 1, x_3 = 0, x_5 = 0$ $\quad n_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

n_2: $x_2 + 3x_4 = -2 \Rightarrow x_2 = -2$; $x_1 = 0, x_3 = 1, x_5 = 0$; $n_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\qquad x_4 = 0$

n_3: $x_2 + 3x_4 = -4 \Rightarrow x_2 = 2$ $\qquad n_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$\qquad x_4 = -2$

$$N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$A^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{pmatrix}$$

~~$L_1 A^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow{P_{34}} P_{34} L_1 A^T =$~~

~~$A^T = \begin{pmatrix} 4 & 6 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $L_1 P_{14} A^T = \begin{pmatrix} 4 & 6 & 2 \\ 0 & & \end{pmatrix}$~~