Matrix Theory, (Fall 1990)

Assignment 7

PART II

Due: Mon., Oct. 15
$\langle\langle$ Householder reflections $\rangle\rangle$

1. If w is a unit vector show that the matrix $Q = I - 2ww^T$ is orthogonal.
2. Describe geometrically the action of Q on an arbitrary vector $u \in R^n$. (Note that the transformation is a reflection about the median of an appropriately defined isosceles triangle).
3. Given a vector $a_1 \in R^n$ use the results in (1)–(2) to construct an orthogonal matrix Q that maps a_1 int a vector b_1 whose only nonzero component is the first, b_{11}.
(note: since $|b_{11}|^2 = \sum a_{1i}^2$, chose $b_{11} = -\frac{a_{11}}{|a_{11}|} \parallel a_1 \parallel$).
That is, show that
$$w_{11} = -\frac{a_{11}}{\parallel a_{11} \parallel}\left(\frac{|a_{11}| + \parallel a_1 \parallel}{S}\right)$$
$$w_{1i} = -\frac{a_{1i}}{S} \quad , \quad i \neq 1,$$
with $S^2 = 2 \parallel a_1 \parallel (|a_{11}| + \parallel a_1 \parallel)$.
(you must show that $S^2 = \parallel b_1 - a_1 \parallel$).
4. Also, construct an orthogonal matrix Q_l that transforms a vector a_l into a vector b_l such that
$$\begin{aligned} b_{li} &= a_{li} \quad , \quad i = 1, \cdots, l-1 \\ b_{li} &= 0 \quad , \quad i = l+1, \cdots, n. \end{aligned}$$
What must b_{ll} be? Choose its sign in accordance to your choice for b_{11}.
5. Utilizing the above ideas, describe an algorithm that accomplishes a factorization $A = QR$ where A is an $m \times n$ $(m \geq n)$ matrix of rank n.
6. How many operations does the algorithm in (6) require?
7. For the same matrix A estimate the operation count for a QR factorization using the Gram–Schmidt procedure. Compare the two methods.