MA464, Matrix Theory, Fall 1999

Examination 1

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Instructions: This exam is due in class Tuesday, Oct. 26. You can use the textbook (or other texts if you want), your notes and your wits. You should not discuss the problems with anyone else, in or outside the class (all problems, hard and easy, are weighted equally).

1. (Graduate students only!!!)
   $\langle\langle$ Householder reflections $\rangle\rangle$

   (a) If w is a unit vector show that the matrix $Q = I - 2ww^T$ is orthogonal.

   (b) Describe geometrically the action of Q on an arbitrary vector $u \in R^n$. (Note that the transformation is a reflection about the median of an appropriately defined isosceles triangle).

   (c) Given a vector $a_1 \in R^n$ use the results in (1)–(2) to construct an orthogonal matrix Q that maps a_1 into a vector b_1 whose only nonzero component is the first, b_{11}.
   Note: since $|b_{11}|^2 = \sum a_{1i}^2$, choose

   $$b_{11} = \begin{cases} -\frac{a_{11}}{|a_{11}|} \parallel a_1 \parallel, & a_{11} \neq 0 \\ - \parallel a_1 \parallel, & a_{11} = 0 \end{cases}$$

   That is, show that

   $$w_{11} = -\frac{a_{11}}{|a_{11}|}\left(\frac{|a_{11}| + \parallel a_1 \parallel}{S}\right)$$

   $$w_{i1} = -\frac{a_{i1}}{S} \quad , \quad i \neq 1,$$

   with $S^2 = 2 \parallel a_1 \parallel (|a_{11}| + \parallel a_1 \parallel)$.
   (you must show that $S^2 = \parallel b_1 - a_1 \parallel^2$).

   (d) Also, construct an orthogonal matrix Q_l that transforms a vector a_l into a vector b_l such that

   $$\begin{aligned} b_{il} &= a_{il} \quad , \quad i = 1, \cdots, l-1 \\ b_{il} &= 0 \quad , \quad i = l+1, \cdots, n. \end{aligned}$$

   What must b_{ll} be? Choose its sign in accordance to your choice for b_{11}.

   (e) Utilizing the above ideas, describe an algorithm that accomplishes a factorization $A = QR$ where A is an $m \times n$ ($m \geq n$) matrix of rank n.

   (f) How many operations does the algorithm in (e) require?

   (g) For the same matrix A estimate the operation count for a QR factorization using the Gramm–Schmidt procedure. Compare the two methods.

2. (Undergraduates only!)
Construct a Householder transformation to map the vector $\mathbf{u}$ with $u^T = (1 \quad 2 \quad 3 \quad 4)$ onto a vector along the x_2 axis. (Two choices).

3. (a) Show that if the $n \times n$ matrix A is symmetric with positive pivots, then there exists a lower triangular $n \times n$ matrix L with positive diagonal entries such that $A = LL^T$ (Cholesky factorization).
   (b) Find the Cholesky factorization for the matrix
$$A = \begin{bmatrix} 4 & 12 \\ 12 & 45 \end{bmatrix}.$$
   (c) Employ the Cholesky factorization to find a least squares solution to the inconsistent system
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

4. A Givens rotation is an orthogonal matrix of the form
$$J_{ik}(\theta) = \begin{array}{c} \begin{bmatrix} 1 & \vdots & & \vdots & \\ \cdots & c & \cdots & s & \cdots \\ & \vdots & & \vdots & \\ \cdots & -s & \cdots & c & \cdots \\ & \vdots & & \vdots & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ i \\ \\ k \\ \\ \end{array}.$$
(Here entries not shown are zero, except for the main diag. which contains 1's in places not shown).
   (a) Show that one can write c and s as $c = \cos\theta$ and $s = \sin\theta$ for some angle θ.
   (b) What is the geometric effect of a Givens rotation $J_{ik}(\theta)$ applied to $\mathcal{R}^n$?
   (c) Show that one can choose a Givens rotation to map the vector $x = (1, 2, 3, 4)^T$ into a vector y of the form $y = (--, --, --, 0)^T$.
   (d) Show how one can choose a Givens rotation R to accomplish the transformation
$$R^T \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 1 \\ 2 & 1 & 8 \end{bmatrix} R = \begin{bmatrix} 4 & 2\sqrt{5} & 0 \\ 2\sqrt{5} & 5.6 & 2.2 \\ 0 & 2.2 & 6.4 \end{bmatrix}.$$
   (Hint: rotate on the y-z plane!).

5. (The Sherman-Morrison-Woodbury formula)
Let A be $n \times n$, nonsingular. Show that if the $m \times m$ matrix T defined by
$$T = I + V^T A^{-1} U$$
is nonsingular, then
$$(A + UV^T)^{-1} = A^{-1} - A^{-1} U T^{-1} V^T A^{-1}.$$

6. Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & -5 \\ -1 & 3 & 9 \end{pmatrix}.$$

(a) Find a basis for $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A^T)$, $\mathcal{R}(A^T)$.

(b) Show explicitly that $\mathcal{R}(A) = \mathcal{N}^{\perp}(A^T)$ and $\mathcal{N}(A) = \mathcal{R}^{\perp}(A^T)$

(c) Find all <u>possible</u> solutions of $Ax = b$ for $b^T = (2 \quad -1 \quad 5 \quad -7)$ and $b^T = (1 \quad 0 \quad 1 \quad 1)$

(d) By identifying the basic and free variables in the above problem, rewrite in the form

$$A'x' = (a_i a_j) \begin{pmatrix} x_i \\ x_j \end{pmatrix} = b - a_k x_k = b'$$

with a_i, a_j the (linearly independent) columns corresponding to the base variables (x_i and x_j respectively), and a_k the column of A corresponding to the free variable x_k. (i, j, and k must be determined!). This is a least squares problem. Solve it by any method and determine all possible answers.

(e) Interpret your results in the previous question geometrically. That is, what is the error $e' = A'x' - b'$ and how does it compare to the error in terms of the original problem $e = Ax - b$?

7. (Projection vs. QR-factorization for the LS problem) Consider the LS problem

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -1 \\ -1 & 3 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Find the Least squares solution:

(a) By using the projection method.

(b) By using QR-factorization, either via Gramm-Schmidt or via Householder transformations.

(c) Are the answers to the previous two problems the same? Why? (or why not?).