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Jens Lorenz

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Department of Mathematics and Statistics,
UNM, Albuquerque, NM 87131

Contents

1 Complex Numbers

1.1 Cartesian Form

Every complex number z can be written as

$$z = x + iy$$

where $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$ are uniquely determined real numbers. The number $z = x + iy$ can be represented in the Cartesian plane by the point with coordinates (x, y).

The set of all complex numbers is denote by $\mathbb{C}$. We are familiar with addition, subtraction, multiplication, and division in $\mathbb{C}$. The usual rules of arithmetic hold. We note that, of course,

$$i^2 = -1 \ .$$

For example,

$$(3 + 4i)(1 + i) = 3 + 4i + 3i - 4 = -1 + 7i \ .$$

1.2 Polar Form

If $z = x + iy$ is a complex number, then

$$r = |z| = \sqrt{x^2 + y^2}$$

is its abolute value. Geometrically, $|z|$ is the distance of z from the origin, and $|z_1 - z_2|$ is the distance between the two complex numbers z_1 and z_2.

Let $z = x + iy$ be a complex number, $z \neq 0$, and let $r = |z|$. From trigonometric geometry we know that there is a unique angle θ with

$$-\pi < \theta \leq \pi$$

and

$$x = r \cos \theta, \quad y = r \sin \theta \ . \tag{1.1}$$

Thus we can write

$$z = r(\cos \theta + i \sin \theta) \ .$$

One calls

$$\theta = \ \mathrm{Arg}\ z$$

the principle argument of z. We have

$$-\pi < \ \mathrm{Arg}\ z \leq \pi \quad \text{for all} \quad z \in \mathbb{C}, \quad z \neq 0 \ .$$

The value $\mathrm{Arg}\ z$ is not defined for $z = 0$.

How can we compute $\mathrm{Arg}\ z$ in a computer code? Let us denote by

$$\arctan\theta, \quad \theta \in \mathbb{R} ,$$

the inverse of the main branch of the tangent function; thus

$$-\frac{\pi}{2} < \arctan\theta < \frac{\pi}{2} .$$

From (1.1) we obtain, for $x \neq 0$,

$$\tan\theta = \frac{y}{x} .$$

Obtain, for $z = x + iy, z \neq 0$,

$$\begin{aligned}
\text{Arg } z &= \arctan\frac{y}{x} \quad \text{for} \quad x > 0 \\
\text{Arg } z &= \arctan\frac{y}{x} + \pi \quad \text{for} \quad x < 0, \quad y > 0 \\
\text{Arg } z &= \arctan\frac{y}{x} - \pi \quad \text{for} \quad x < 0, \quad y < 0 \\
\text{Arg } z &= \frac{\pi}{2} \quad \text{for} \quad x = 0, \quad y > 0 \\
\text{Arg } z &= -\frac{\pi}{2} \quad \text{for} \quad x = 0, \quad y < 0
\end{aligned}$$

Euler's Identity says that

$$\cos\theta + i\sin\theta = e^{i\theta} .$$

We comment on this below.

Simple implications of Euler's identity are:

$$e^{2\pi i} = 1, \quad e^{\pi i} = -1, \quad e^{\pi i/2} = i, \quad e^{3\pi i/2} = -i .$$

Using Euler's identity, the polar form of $z = x + iy$ is

$$z = re^{i\theta}$$

where

$$r = |z|, \quad \theta = \text{ Arg } z .$$

Application of Euler's Identity: The identity is very useful for computing roots of complex numbers. For example, let

$$z = 1 + i = \sqrt{2}\, e^{i\pi/4} .$$

There are three numbers z_1, z_2, z_3 with $z_j^3 = z$. These are

$$\begin{aligned}
z_1 &= 2^{1/6} e^{\pi i/12} \\
z_2 &= 2^{1/6} e^{\pi i/12} e^{2\pi i/3} \\
z_3 &= 2^{1/6} e^{\pi i/12} e^{4\pi i/3}
\end{aligned}$$

1.3 Complex Conjugates

If $z = x + iy$ is a complex number, where x and y are real, then $\bar{z} = x - iy$ denotes the complex conjugate of z. Geometrically, $\bar{z}$ is obtained form $z$by reflection about the x–axis.

The rules

$$\begin{aligned} \overline{z+w} &= \bar{z} + \bar{w} \\ \overline{zw} &= \bar{z}\bar{w} \\ \overline{1/z} &= 1/\bar{z} \end{aligned}$$

are useful.

1.4 Remarks on Euler's Identity

From calculus, we recall the Taylor series of a smooth function $f(x)$. If $x \sim x_0$ then

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \ldots$$

The power series

$$\sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(x_0)(x - x_0)^j$$

is called the Taylor series of the function $f(x)$ about x_0.

Example: Let $f(x) = e^x, x_0 = 0$. We have $f^{(j)}(x) = e^x$ for $j = 1, 2, \ldots$ and $f^{(j)}(0) = 1$. Here:

$$e^x = \sum_{j=0}^{\infty} \frac{1}{j!} x^j \ .$$

Motivated by the Taylor expansions of the real functions $e^x = \exp x, \cos x, \sin x$ we define for complex z:

$$\begin{aligned} \exp z &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \\ \sin z &= \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!} \\ \cos z &= \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!} \end{aligned}$$

One can then show that

$$\exp(iz) = \cos z + i \sin z \tag{1.2}$$

and

$$\exp(a+b) = \exp(a)\exp(b)\ . \tag{1.3}$$

The equation (1.2) is Euler's identity for complex arguments. The equation (1.3) is the fundamental law for the exponential function.

Proof of (1.2):

$$\begin{aligned}
\exp(iz) &= 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots \\
&= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \dots + i\Big(\frac{z}{1!} - \frac{z^3}{3!} + \dots\Big) \\
&= \cos z + i \sin z
\end{aligned}$$

The following may make the identity (1.3) plausible: We have

$$\begin{aligned}
\exp(a+b) &= 1 + \frac{a+b}{1!} + \frac{(a+b)^2}{2!} + \frac{(a+b)^3}{3!} + \dots \\
&= 1 + a + b + \frac{1}{2}(a^2 + 2ab + b^2) + \frac{1}{3!}(a^3 + 3a^2b + 3ab^3 + b^3) + \dots
\end{aligned}$$

and

$$\begin{aligned}
\exp(a)\exp(b) &= \Big(1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \dots\Big) \cdot \Big(1 + b + \frac{b^2}{2} + \frac{b^3}{3!} + \dots\Big) \\
&= 1 + a + b + \frac{1}{2}(a^2 + 2ab + b^2) + \frac{1}{3!}(a^3 + 3a^2b + 3ab^2 + b^3) + \dots
\end{aligned}$$

1.5 Application of Euler's Identity: De Moivre's Formula

De Moivre's formula expresses $\cos n\theta$ as a polynomial in $\cos\theta$. Its derivation is a good application of Euler's identity.

Let us recall the binomial coefficients

$$\left(\begin{array}{c} n \\ j \end{array}\right) = \frac{n!}{j!(n-j)!}$$

and the binomial formula

$$(a+b)^n = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array}\right) a^{n-j} b^j\ .$$

Abbreviation:

$$c = \cos\theta, \quad s = \sin\theta$$

We have

$$
\begin{aligned}
\cos n\theta + i\sin n\theta &= e^{in\theta} \\
&= \left(e^{i\theta}\right)^n \\
&= (c+is)^n \\
&= \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) c^{n-j} i^j s^j
\end{aligned}
$$

Let $n = 3$, for example. Obtain:

$$\cos 3\theta + i\sin 3\theta = c^3 + 3ic^2 s - 3cs^2 - is^3$$

Taking real parts:

$$
\begin{aligned}
\cos 3\theta &= c^3 - 3cs^2 \\
&= c^3 - 3c(1-c^2) \\
&= -3c + 4c^3 \\
&= -3\cos\theta + 4\cos^3\theta
\end{aligned}
$$

The polynomial

$$T_3(x) = -3x + 4x^3$$

is the 3rd Chebyshev polynomial and we have obtained:

$$\cos 3\theta = T_3(\cos\theta) \ .$$

Notation: If q is any real number, let $[q]$ denote its integer part and let $\{q\}$ denote its fractional part, i.e,

$$q = [q] + \{q\}, \quad [q] \in \mathbb{Z}, \quad 0 \le \{q\} < 1 \ .$$

We have

$$\cos n\theta = \mathrm{Re}\left(\sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) c^{n-j} i^j s^j \right)$$

Here i^j is real if and only if $j = 2k$ is even, $0 \le k \le [n/2]$. Obtain:

$$\cos n\theta = \sum_{k=0}^{[n/2]} (-1)^k \left(\begin{array}{c} n \\ 2k \end{array} \right) \cos^{n-2k}\theta \left(1 - \cos^2\theta\right)^k \ .$$

We have obtained that

$$\cos n\theta = T_n(\cos\theta) \tag{1.4}$$

where

$$T_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} x^{n-2k}(1-x^2)^k$$

is the n–th Chebyshev polynomial.

A short list:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= -1 + 2x^2 \\ T_3(x) &= -3x + 4x^3 \end{aligned}$$

One can use the identity (1.4) to obtain information about the polynomials $T_n(x)$, which have degree n. If θ varies in the interval $0 \le \theta \le \pi$, the variable $x = \cos\theta$ varies once from $x = 1$ to $x = -1$. It follows that

$$|T_n(x)| \le 1 \quad \text{for} \quad -1 \le x \le 1$$

and $T_n(x)$ has n zeros in the open interval $-1 < x < 1$ since $\cos n\theta$ vanishes n times for $0 \le \theta \le \pi$.

1.6 The Principle Branch of the Complex Logarithm

If $r > 0$ is a positive real number, let $\ln r$ denote its natural logarithm. Thus,

$$e^{\ln r} = r, \quad r > 0 \ .$$

One wants to extend the natural logarithm into the complex plane (more precisely, into $\mathbb{C} \setminus \{0\}$) and obtain a function $\log z$ which inverts the exponential function, i.e,

$$e^{\log z} = z, \quad z \ne 0 \ .$$

This can be done as follows: For $z \ne 0$ let

$$z = re^{i\theta}$$

with

$$r = |z| > 0$$

and

$$-\pi < \theta = \text{ Arg } z \le \pi \ .$$

Since

$$r = e^{\ln r}$$

we have

$$z = e^{\ln r} e^{i\theta} = e^{\ln r + i\theta} \ .$$

Therefore, one defines for $z \neq 0$,

$$\log z = \ln |z| + i \ \mathrm{Arg}\ z.$$

The function $\log z$ extends the real function $\ln r$ into $\mathbb{C} \setminus \{0\}$ and satisfies

$$z = e^{\log z}, \quad z \neq 0 \ .$$

The function $\log z$ is called the principle branch of the complex logarithm function. It is important to note that the function $\log z$ is discontinuous along the negative real axis.

One can use $\log z$ to define general powers. Let z and a denote complex numbers, $z \neq 0$. Since

$$z = e^{\log z}$$

one defines

$$z^a = \exp(a \log z)$$

as the principle branch of z^a.

2 Infinite Series

2.1 Convergence of Sequences

Let $z_1, z_2, \ldots$ denote a sequence of complex numbers. We often simply write z_n for a sequence. A fundamental concept of mathematics is *convergence* of a sequence, which is defined as follows:

Definition: If z_n is a sequence in $\mathbb{C}$ and if $z \in \mathbb{C}$, then

$$z_n \to z \quad \text{as} \quad n \to \infty$$

means: For every $\varepsilon > 0$ there is $N \in \mathbb{N}$ with $|z_n - z| < \varepsilon$ for all $n \geq N$.

One hardly ever works directly with this definition, except for very simple sequences.

Examples: Use the definition to prove that

$$\frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty$$

and if $|a| < 1$ then

$$a^n \to 0 \quad \text{as} \quad n \to \infty \ .$$

A more complicated example of convergence is

$$z_n = \frac{n^2 + n + 3}{2n^2 + 5} = \frac{1 + \frac{1}{n} + \frac{3}{n^2}}{2 + \frac{5}{n^2}} \to \frac{1}{2} \quad \text{as} \quad n \to \infty \ .$$

The following two results are theoretically very important since, at their heart, lies completeness of the fields of complex and real numbers.

Cauchy Criterion: A sequence of complex numbers z_n converges if, given any $\varepsilon > 0$, there is $N \in \mathbb{N}$ with

$$|z_k - z_n| < \varepsilon \quad \text{for} \quad k, n \geq N \ .$$

Monotone Bounded Sequences Converge: If s_n is a bounded, monotone sequence of real numbers, then s_n converges.

Example: Definition of Euler's constant:

$$\gamma = \lim_{n \to \infty} \Big(\sum_{j=1}^{n} \frac{1}{j} - \ln n \Big)$$

Let

$$s_n = \sum_{j=1}^{n} \frac{1}{j} - \ln n \ .$$

For example,

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \int_1^4 \frac{dx}{x} \ .$$

This can be illustrated by an area.

We have

$$s_n - s_{n+1} = -\frac{1}{n+1} + \int_n^{n+1} \frac{dx}{x} \ ,$$

thus

$$s_n > s_{n+1} \ .$$

Also, by considering an area, the inequality

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \geq \int_2^n \frac{dx}{x} = \ln n - \ln 2$$

follows. It implies that

$s_n \geq 1 - \ln 2 > 0$. Thus, the sequence s_n decreases monotonically and is bounded from below. Therefore,

$$\lim s_n =: \gamma$$

exists. One can compute

$$\gamma \sim 0.577216\ldots$$

It is not known if γ is rational or irrational.

2.2 Convergence of Series

In applications, sequences z_n often occur as partial sums of series.

If $u_0, u_1, \ldots$ is a sequence, then the expression

$$\sum_{j=0}^{\infty} u_j$$

is called a series. Its n–th partial sum is

$$z_n = \sum_{j=0}^{n} u_j \ .$$

By definition, the series $\sum u_j$ converges if its sequence of partial sums converges. If

$$z_n = \sum_{j=0}^{n} u_j \to z \quad \text{as} \quad n \to \infty$$

then one also writes

$$\sum_{j=0}^{\infty} u_j = z \ .$$

Thus, the symbol

$$\sum_{j=0}^\infty u_j$$

typically has two meanings. It can denote an expression or, if the series converges, it also stands for the limit of its partial sums.

Example: The series

$$\sum_{j=1}^\infty j$$

diverges. It is less obvious that the harmonic series

$$\sum_{j=1}^\infty \frac{1}{j}$$

also diverges. We will show this below using the integral test.

Example: Let $a \in \mathbb{C}$ and consider the geometric series

$$\sum_{j=0}^\infty a^j \ .$$

If $a \neq 1$ then its n–th patial sum is

$$s_n = 1 + a + a^2 + \ldots + a^n = \frac{1-a^{n+1}}{1-a} \ .$$

Thus, the geometric series converges if $|a| < 1$ and

$$\sum_{j=0}^\infty a^j = \frac{1}{1-a} \quad \text{if} \quad |a| < 1 \ .$$

Recall the most important convergence test for series:

Comparison test: Let $b_j \geq 0$ and assume that $\sum_j b_j$ converges. If $|z_j| \leq b_j$ for all j, then $\sum_j z_j$ also converges. The proof uses the Cauchy criterion.

Ratio test: Assume that $z_j \neq 0$ for all j. Assume the limit

$$\lim_{j\to\infty} \frac{|z_{j+1}|}{|z_j|} =: r$$

exists. If $0 \leq r < 1$, then the series $\sum_j z_j$ converges. If $1 < r \leq \infty$, then the series $\sum_j z_j$ diverges.

Proof: a) Let $0 \leq r < q < 1$. There is J with

$$|z_{j+1}| \leq q|z_j| \quad \text{for} \quad j \geq J \ .$$

This yields

$$|z_{j+k}| \leq |z_J| q^k, \quad k = 0, 1, \ldots$$

Convergence of $\sum_j z_j$ then follows from $\sum q^k < \infty$ by the comparison test.

b) Let $1 < q < r$. We have

$$|z_{j+1}| \geq q|z_j| \quad \text{for} \quad j \geq J \ ,$$

and $|z_j| \to \infty$ for $j \to \infty$.

Examples: Use the ratio test to show that the series defining $e^z, \cos z, \sin z$ converge for every $z \in \mathbb{C}$.

Integral tests: We illustrate convergence and divergence of series by two examples.

Example: We know from analysis that

$$\int_1^\infty \frac{dx}{x^\lambda} < \infty \quad \text{if} \quad \lambda > 1$$

and

$$\int_1^\infty \frac{dx}{x} = \infty \ .$$

Consider Riemann's zeta function:

$$\zeta(\lambda) = \sum_{j=1}^\infty \frac{1}{j^\lambda} \ .$$

Comparison with the corresponding integral implies that $\zeta(\lambda)$ is finite for $\lambda > 1$, but has a singularity at $\lambda = 1$.

2.3 Alternating Series

The alternating harmonic series converges:

$$\sum_{j=1}^\infty (-1)^{j+1} \frac{1}{j} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$$

To see this, one can use **Leibniz criterion:** Assume that

$$a_j \geq a_{j+1} > 0 \quad \text{for all} \quad j$$

and

$$a_j \to 0 \quad \text{as} \quad j \to \infty \ .$$

Then

$$\sum_{j=1}^\infty (-1)^{j+1} a_j$$

converges. Proof: Consider

$$s_{2n} = a_1 + (-a_2 + a_3) + \ldots + (-a_{2n-2} + a_{2n-1}) - a_{2n} \ .$$

It is clear that $s_{2n} \leq a_1$.

Also,

$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n} \ .$$

Thus, $s_{2n} \to s$, monotonically increasing. Furthermore, since $a_{2n+1} \to 0$,

$$s_{2n+1} = s_{2n} + a_{2n+1} \to s \ .$$

This yields that $s_n \to s$.

A series $\sum_j z_j$ of complex numbers is said to converge absolutely, if the series of absolute values,

$$\sum_j |z_j|$$

converges. By the comparison test, any absolutely convergent series converges. The converse is not true, as is shown by the alternating harmonic series. This series converges, but does not converge absolutely.

A convergent series, which does not converge absolutely, is said to converge conditionally.

Operations like reordering of terms or inserting brackets are allowed for absolutely convergent series. These operations then do not change the value of the series. On the other hand, if a series converges only conditionally, reordering typically changes the value of the series.

2.4 Series of Functions

In applications, series often contain a variable. A standard example is the exponential series,

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \quad z \in \mathbb{C} \ .$$

The most important modes of convergence for a series of functions are pointwise convergence and uniform convergence.

We explain these concepts again first for a sequence of functions.

2.4.1 Pointwise and Uniform Convergence

Let $\Omega \subset \mathbb{C}$ and let $s_n(z), z \in \Omega$, denote a sequence of functions. The sequence converges pointwise on Ω to $s(z)$ if for every $z \in \Omega$ and for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ with

$$|s_n(z) - s(z)| < \varepsilon \quad \text{for} \quad n \geq N \ .$$

It is important to note that N is allowed to depend on z. Of course, N will also depend on ε.

The sequence $s_n(z)$ converges uniformly on Ω to $s(z)$ if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ with

$$|s_n(z) - s(z)| < \varepsilon \quad \text{for} \quad n \geq N \quad \text{and} \quad z \in \Omega \ .$$

In this case, N is not allowed to depend on z. One can say that the error $|s_n(z)-s(z)|$ converges to zero uniformly on Ω, at a speed that does not depend on z.

The distinction between pointwise and uniform convergence may seem rather minor. It is crucial, however, if one wants to operate on a convergent sequence, like integrate it.

If $\sum_{j=0}^{\infty} u_j(z)$ is a series of functions, then one considers the sequence of partial sums,

$$s_n(z) = \sum_{j=0}^{n} u_j(z), \quad z \in \Omega \ .$$

Pointwise and uniform convergence for the series is then defined in terms of the sequence of partial sums.

Examples: 1) $s_n(z) = z^n$. The sequence converges pointwise for $|z| < 1$ and for $z = 1$. The pointwise limit is the function

$$s(z) = 0 \quad \text{for} \quad |z| < 1, \quad s(1) = 1 \ .$$

We see here that pointwise limit of smooth functions may be discontinuous. Here one cannot exchange limits: Let

$$z_j = 1 - \frac{1}{j}, \quad j = 1, 2, \ldots$$

We have

$$s_n(z_j) = \Big(1 - \frac{1}{j}\Big)^n$$

and

$$\lim_{n\to\infty} \lim_{j\to\infty} s_n(z_j) = 1$$

and

$$\lim_{j\to\infty} \lim_{n\to\infty} s_n(z_j) = 0 \ .$$

This example shows that one may not be allowed to exchange the order of limits when one deals with non–uniform convergence.

2) Consider

$$\sum_{j=0}^{\infty} (1-x)x^j, \quad 0 \le x \le 1 \ .$$

The series converges pointwise to

$$s(x) = 1 \quad \text{for} \quad 0 \le x < 1, \quad s(1) = 0 \ .$$

Again, we note that the pointwise limit of continuous functions can be discontinuous.

3) Let $s_n(x), 0 \le x \le 1$, be the piecewise linear function with

$$s_n(0) = 0 = s(\frac{2}{n}), \quad s_n(\frac{1}{n}) = n, \quad s_n(x) = 0 \quad \text{for} \quad x > \frac{2}{n} \ .$$

The sequence converges to 0 pointwise. We have

$$\int_0^1 s_n(x)\,dx = 1 \quad \text{for all} \quad n \ .$$

The limit function has the integral 0.

The examples show the difficulties that may occur if convergence is not uniform. On the positive side, the following holds:

Theorem 2.1 *a) The uniform limit of continuous functions is continuous. b) If $s_n(x) \to s(x)$ uniformly for $a \le x \le b$ and if $s_n(x)$ is continuous for every n, then*

$$\int_a^b s_n(x)dx \to \int_0^b s(x)dx \ .$$

2.4.2 The Weierstrass' $M-$Test

A very useful and simple result on the convergence of a series of functions is the so–called Weierstrass' M-test: Let $\Omega \subset \mathbb{C}$ and let

$$u_j(z), z \in \Omega, \quad j = 0, 1, \ldots$$

denote a sequence of functions. We want to study the series

$$\sum_{j=0}^{\infty} u_j(z) \ .$$

Suppose that

$$|u_j(z)| \le M_j \quad \text{for all} \quad z \in \Omega$$

and that

$$\sum_j M_j < \infty \ .$$

Then the series $\sum_j u_j(z)$ converges absolutely for every $z \in \Omega$ and the convergence is uniform on Ω. In particular, if every function $u_j(z)$ is continuous on Ω, then the series $\sum_j u_j(z)$ defines a continuous function on Ω.

2.4.3 Power Series:

The radius of convergence of the power series

$$\sum_{j=0}^{\infty} a_j z^j$$

is $R = \frac{1}{L}$ if

$$\lim_{j\to\infty} \left| a_{j+1}/a_j \right| = L \tag{2.1}$$

exists. Assume the limit exists. Then the function

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad |z| < R \ ,$$

is continuous. In fact, $f(z)$ is complex differentiable and the derivative can be obtained by termwise differentiation. The series

$$f'(z) = \sum_{j=1}^{\infty} a_j j x^{j-1}$$

also has radius of convergence R.

2.4.4 Taylor Series With Remainder

Recall Taylor expansion with remainder about the point $x_0 = a$: If $f \in C^{n+1}[a,b]$ then

$$f(x) = p_n(x) + R_n(x)$$

where

$$p_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!} (x-a)^j$$

and

$$\begin{aligned} R_n(x) &= \frac{1}{n!} \int_a^x (x-s)^n f^{(n+1)}(s)\, ds \\ &= \frac{1}{(n+1)!}(x-a)^{n+1} f^{(n+1)}(\xi), \quad a \le \xi \le x \ . \end{aligned}$$

In the last equation, we have used a mean value theorem for integrals that we show below.

Proof of the formula $f(x) = p_n(x) + R_n(x)$ through integration by parts:

$$\begin{aligned} f(x) - f(a) &= \int f'(s)\, ds \\ &= (s-x)f'(s)|_a^x - \int_a^x (s-x)f''(s)\, ds \\ &= (x-a)f'(a) - \int_a^x (s-x)f''(s)\, ds \\ &= (x-a)f'(a) - \frac{1}{2}(s-x)^2 f''(s)|_a^x + \frac{1}{2}\int_a^x (s-x)^2 f'''(s)\, ds \\ &= (x-a)f'(a) + \frac{1}{2}(x-a))^2 f''(a) + \frac{1}{2}\int_a^x (s-x)^2 f'''(s)\, ds \end{aligned}$$

etc.

Example 1: Define $f(x)$ as the solution of the initial value problem

$$f' = f, \quad f(0) = 1 \ .$$

One obtains that the Taylor series of $f(x)$ is

$$\sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x \ .$$

In fact, we know that $f(x) = e^x$, i.e., $f(x)$ is given by its Taylor series,

$$f(x) = e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \ .$$

Example 2: Let $f(x) = \ln(1+x), x > -1$. The Taylor series about $a = 0$ is

$$\sum_{j=1}^{\infty} \frac{1}{j} (-1)^{j+1} x^j \ .$$

Computation:

$$f^{(j)}(x) = (-1)^{j+1}(j-1)!(1+x)^{-j}, \quad j \geq 1 \ .$$

Using the ratio test it follows that the Taylor series converges for $-1 < x < 1$. The Taylor series also converges for $x = 1$, but not for $x = -1$.

The Taylor series does not converge for $|x| > 1$. Estimating the remainder, it follows that the Taylor series converges to $f(x) = \ln(1+x)$ for $0 \leq x \leq 1$:

Estimate of the remainder: We have, with some ξ in the interval $0 \leq \xi \leq x$:

$$\begin{aligned} |R_n(x)| &= \frac{x^{n+1}}{(n+1)!} |f^{(n+1)}(\xi)| \\ &= \frac{1}{n+1} \Big(\frac{x}{1+\xi} \Big)^{n+1} \end{aligned}$$

This shows that $|R_n(x)| \to 0$ as $n \to \infty$. In other words:

$$\ln(1+x) = \sum_{j=1}^{\infty} \frac{1}{j} (-1)^{j+1} x^j \quad \text{for} \quad 0 \leq x \leq 1 \ .$$

In particular,

$$\sum_{j=1}^{\infty} \frac{1}{j} (-1)^{j+1} = \ln 2 \ .$$

Remark: For $z \in \mathbb{C}$ with $|z| < 1$ one can show that

$$\log(1+z) = \sum_{j=1}^{\infty} \frac{1}{j} (-1)^{j+1} z^j, \quad |z| < 1 \ .$$

This follows since the left–hand side and the right–hand side are analytic functions of z for $|z| < 1$ that agree for $z = x$ with $0 \le x < 1$. The identity theorem for analytic functions then leads to the above equation.

Example 3: $f(x) = \frac{1}{1+x^2}$ The Taylor series is

$$\sum_{j=0}^{\infty} (-1)^j x^{2j} \ .$$

By the ratio test, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

2.5 Asymptotic Series

Let us illustrate the concept of an asymptotic series by an example. We want to know how the function

$$I(x) = \int_x^\infty e^{-u} u^{-2} \, du, \quad x > 0 \ ,$$

'behaves' for $x \to \infty$. This means that we want to approximate $I(x)$, for large x, by a simple expression that we can comprehend. If possible, we also want to have an estimate of the error between $I(x)$ and the simple expression. For the above integral it is a good idea to use integration by parts since, if we differentiate u^{-2}, the derivative $-2u^{-3}$ decays faster than u^{-2}.

We obtain through integration by parts:

$$\begin{aligned} I(x) &= \int_x^\infty e^{-u} u^{-2} \, du \\ &= -e^{-u} u^{-2} \Big|_x^\infty - 2 \int_x^\infty e^{-u} u^{-3} \, du \\ &= \frac{e^x}{x^2} + R_0(x) \end{aligned}$$

Here e^x/x^2 is the simple expression approximating $I(x)$, and the remainder is

$$R_0(x) = -2 \int_x^\infty e^{-u} u^{-3} \, du \ .$$

Using the simple substitution $u = x + v$ we have

$$\begin{aligned} \int_x^\infty e^{-u} u^{-3} \, du &= \int_0^\infty e^{-x-v} \frac{1}{(x+v)^3} \, dv \\ &= \frac{e^{-x}}{x^3} \int_0^\infty \frac{e^{-v}}{(1+v/x)^3} \, dv \end{aligned}$$

This shows that

$$|R_0(x)| \le 2 \, \frac{e^{-x}}{x^3} \ .$$

To summaraize, we have obtained that

$$I(x) = \frac{e^{-x}}{x^2} + R_0(x) \quad \text{where} \quad |R_0(x)| \le 2\,\frac{e^{-x}}{x^3}\ .$$

This is a good result since e^{-x}/x^2 is a simple expression and the remainder $R_0(x)$ decays faster to zero as $x \to \infty$ than e^{-x}/x^2.

One can now improve the approximation by using integration by parts for $R_0(x)$:

$$\begin{aligned} R_0(x) &= -2\int_x^\infty e^{-u}u^{-3}\,du \\ &= -2e^{-u}/u^3 + R_1(x) \end{aligned}$$

with

$$R_1(x) = 2\cdot 3\int_x^\infty e^{-u}u^{-4}\,du$$

This yields the improved approximation of $I(x)$:

$$I(x) = \frac{e^{-x}}{x^2} - 2\frac{e^{-x}}{x^3} + R_1(x)$$

with

$$|R_1(x)| \le 6e^{-x}x^{-4}\ .$$

If one repeats the process of integrating the remainder by parts, one obtains that for any $n = 0, 1, \ldots$:

$$I(x) = e^{-x}\Big(\frac{1}{x^2} - \frac{2}{x^3} + \frac{6}{x^4} - \ldots + (-1)^n\frac{(n+1)!}{x^{n+2}}\Big) + R_n(x)$$

with

$$\begin{aligned} |R_n(x)| &= (n+2)!\int_x^\infty e^{-u}u^{-n-3}\,du \\ &= (n+2)!e^{-x}x^{-n-3}\int_0^\infty e^{-v}\Big(1+\frac{v}{x}\Big)^{-n-3}dv \\ &\le (n+2)!e^{-x}x^{-n-3}\ . \end{aligned}$$

In the formula

$$I(x) = e^{-x}\Big(\sum_{j=0}^{n}(j+1)!(-1)^j x^{-j-2}\Big) + R_n(x)$$

the terms in the sum decay faster with increasing j and the remainder decays faster than the last term in the sum. In some sense, the approximation of $I(x)$ by the finite sum

$$e^{-x}\Big(\sum_{j=0}^{n}(j+1)!(-1)^j x^{-j-2}\Big)$$

gets better and better with increasing n since the remainder, $R_n(x)$, decays faster and faster with increasing n.

It is now tempting to let $n \to \infty$. However, the series

$$e^{-x}\Big(\sum_{j=0}^{\infty}(j+1)!(-1)^j x^{-j-2}\Big)$$

does not converge for any $x > 0$. One says that the above series is an asymptotic approximation of $I(x)$ as $x \to \infty$ and writes

$$I(x) \sim e^{-x}\Big(\sum_{j=0}^{\infty}(j+1)!(-1)^j x^{-j-2}\Big) \quad \text{as} \quad x \to \infty \; .$$

This means that, for every fixed n, one has

$$I(x) \sim e^{-x}\Big(\sum_{j=0}^{n}(j+1)!(-1)^j x^{-j-2}\Big) \quad \text{as} \quad x \to \infty$$

where the error decays faster than the last term in the approximation.

Asymptotic expansions into series are often very useful if one wants to obtain insight into the behaviour of a complicated function or expression. This holds true despite the fact that asumptotic series often diverge.

2.6 Supplement 1: Integration by Parts

Let $F, G \in C^1[a,b]$. The product rule says

$$(FG)' = F'G + FG' \; .$$

Integrating over $a \le x \le b$ we find that

$$(FG)|_a^b = \int_a^b F'G\,dx + \int_a^b FG'\,dx \; .$$

We may also write this as

$$\int_a^b F'G\,dx = (FG)|_a^b - \int_a^b FG'\,dx \; .$$

With $f = F'$ this equation becomes:

$$\int_a^b f(x)G(x)\,dx = (FG)|_a^b - \int_a^b F(x)G'(x)\,dx \; .$$

This rule of partial integration is often used when one has to integrate a product of two functions, here $f(x)$ and $G(x)$. One integrates one factor, $f(x)$, determining FG between a and b and subtracts the integral $\int_a^b F(x)G'(x)\,dx$.

2.7 Supplement 2: A Mean Value Theorem for Integrals

Let $f, g \in C[a,b]$ and let

$$g(x) \geq 0 \quad \text{for all} \quad a \leq x \leq b \ .$$

Then there is a number ξ with $a \leq \xi \leq b$ so that

$$\int_a^b f(x)g(x)\,dx = f(\xi) \int_a^b g(x)\,dx \ .$$

Proof: Let

$$m = \min_{a \leq x \leq b} f(x), \quad M = \max_{a \leq x \leq b} f(x) \ .$$

We have

$$mg(x) \leq f(x)g(x) \leq Mg(x)$$

and integration yields

$$mJ \leq \int f(x)g(x)\,dx \leq MJ, \quad J = \int_a^b g(x)\,dx \ .$$

We may assume $J > 0$ and obtain

$$m \leq \frac{1}{J} \leq \int f(x)g(x)\,dx \leq \frac{M}{J} \ .$$

By the intermediate value theorem applied to f, the function f attains every value between m and M. This shows the claim.

3 Functions of a Complex Variable

3.1 Recall: Complex Numbers; the Exponential and the Logarithm Function

Representation $z = x + iy$; addition, multiplication. The complex conjugate $\bar{z} = x - iy$. The absolute value $|z| = \sqrt{x^2 + y^2}$. The distance $|z_1 - z_2|$.

Concept of convergence.

Power series:

$$
\begin{aligned}
e^z = \exp(z) &= \sum_{j=0}^{\infty} \frac{1}{j!}\, z^j \\
\sin(z) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}\, z^{2j+1} \\
\cos(z) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}\, z^{2j}
\end{aligned}
$$

Exponential law:

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

Euler's identity:

$$e^{iz} = \cos z + i \sin z$$

Polar form, for $z \neq 0$:

$$z = re^{i\theta}, \quad r = |z|, \quad -\pi < \theta \le \pi \ .$$

De Moivre's formula.

The principle branch of $\ln z$: Write $z = re^{i\theta}$ as above. Then

$$\ln z = \ln r + i\theta \ .$$

3.2 Complex Differentiability; the Cauchy–Riemann Equations

Notations: If z_0 is a complex number and if $r > 0$ then

$$B_r(z_0) = \{z : |z - z_0| < r\} \ .$$

We write B_r for $B_r(0)$.

A set $\Omega \subset \mathbb{C}$ is called open if for every $z_0 \in \Omega$ there is $r > 0$ with $B_r(z_0) \subset \Omega$. A set $\Omega \subset \mathbb{C}$ is called a region if Ω is open and connected.

Examples of open sets are the open circles $B_r(z_0)$, the set $\mathbb{C}$, the open right half–plane,

$$H^+ = \{z : \operatorname{Re} z > 0\} \ .$$

In the following three definitions, Ω is an open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a complex valued function defined on Ω.

Definition 3.1 The function f is called complex analytic on Ω (or, simply, analytic) if for every $z_0 \in \Omega$ the function $f(z)$ can be written as a power series which converges in some neighborhood $B_r(z_0)$ of z_0:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < r \ .$$

Here $r > 0$ depends on z_0.

Definition 3.2 The function f is called complex differentiable in a point $z_0 \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{1}{h}(f(z_0 + h) - f(z_0))$$

exists. The limit is then denoted by $f'(z_0$ and is called the complex derivative of f at z_0. The function f is called complex differentiable in Ω if it is complex differentiable in every point $z_0 \in \Omega$.

Definition 3.3 Let

$$f(x + iy) = u(x, y) + iv(x, y) \quad \text{for} \quad z = x + iy \in \Omega$$

where u and v are real–valued functions. Then f is called holomorphic in Ω if $u, v \in C^1(\Omega$ and

$$u_x = v_y, \quad u_y = -v_x \quad \text{in} \quad \Omega \ .$$

The above equations are called the Cauchy–Riemann equations for u and v.

An important theorem of complex variables says that the three definitions characterize the same functions $f : \Omega \to \mathbb{C}$.

Theorem 3.1 *For any function $f : \Omega \to \mathbb{C}$ the following three conditions are equivalent:*

1. *f is complex analytic in Ω;*
2. *f is complex differentiable in Ω;*
3. *f is holomorphic in Ω.*

If f satisfies the conditions of the theorem, then we write $f \in H(\Omega)$.

Power series can be differentiated arbitrarily often in their circle of convergence. Therefore, if $f \in H(\Omega)$, then the complex derivatives of any order of f exist on Ω. In particular, $u, v \in C^\infty(\Omega)$ and

$$\Delta u = \Delta v = 0 \ .$$

The real part and the imaginary part of a holomorphic function is a harmonic function.

3.3 Line Integrals; Cauchy's Integral Theorem

Let $\mathcal{C}$ denote a curve parametrized by $\gamma(t), a \le t \le b$. Let f be continuous on $\mathcal{C}$. Then

$$\int_{\mathcal{C}} f(z)dz$$

is defined as a limit of Riemann sums:

$$\sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1}) \; .$$

Here $z_0, z_1, \ldots, z_n$ are consecutive points on $\mathcal{C}$ and ξ_j is a point on $\mathcal{C}$ between z_{j-1} and z_j.

One obtains the simple estimate:

$$|\int_{\mathcal{C}} f(z)dz| \le \max_{z \in \mathcal{C}} |f(z)| * length(\mathcal{C}) \; .$$

One often computes the line integral by using a parametrization of $\mathcal{C}$:

$$\int_{\mathcal{C}} f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

if $\gamma(t), a \le t \le b$, parametrizes $\mathcal{C}$.

Important example:

$$\int_{\mathcal{C}_1} \frac{dz}{z} = 2\pi i$$

where $\mathcal{C}_1$ is the unit circle.

Also, if $n \in \mathbb{Z}, n \ne -1$, then

$$\int_{\mathcal{C}_1} z^n \, dz = 0 \; .$$

The same results hold if $\mathcal{C}_1$ is replaced by the circle of radius $r > 0$.

Cauchy's Integral Theorem:

Theorem 3.2 *Let $f \in H(\Omega)$ and let $\mathcal{C}$ be a closed curve in Ω. Assume the region inside of $\mathcal{C}$ belongs to Ω. Then the line integral of f along $\mathcal{C}$ is zero:*

$$\int_{\mathcal{C}} f(z)dz = 0 \; .$$

Implication: independence of path.

Evaluations of intergals using deformation of the path.

3.4 Cauchy's Integral Formula; Cauchy's Inequalities

Let $\mathcal{C}$ be a closed curve and let f be cdb in the region Ω inside $\mathcal{C}$. For $z \in \Omega$:

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{\xi - z}\, d\xi \; .$$

One can differntiate under the integral sign. Obtain that $f \in C^\infty$ and

$$f^{(j)}(z) = \frac{j!}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^{j+1}}\, d\xi \; .$$

Use this formula for $f^{(j)}(z_0)$ in the Taylor series

$$\sum_{j=0}^\infty \frac{f^{(j)}(z_0)}{j!}\,(z - z_0)^j \; .$$

Use a circle $\mathcal{C}_r$ about z_0 to express $f^{(j)}(z_0)$. Then the Taylor series converges to $f(z)$ for $|z - z_0| < r$.

Let

$$f(z) = \sum_{j=0}^\infty a_j z^j, \quad |z| \le r \; .$$

Let

$$M(r) = max_{|z|=r} |f(z)| \; .$$

Then obtain

$$|a_j| r^j \le M(r) \; .$$

This yields Liouville's theorem, which implies the Fundamental Theorem of Algebra. Also, if $f(z)$ is an entire function with

$$|f(z)| \le C(1 + |z|^N), \quad z \in \mathbb{C} \; ,$$

then $f(z)$ is a polynomial of degree $\le N$.

3.5 Analytic Continuation

Example 1:

$$f(z) = \sum_{j=0}^\infty z^j = \frac{1}{1-z}, \quad |z| < 1 \; .$$

The series defines an analytic function $f(z)$ in the unit ball $\Omega = B_1$. One can continue $f(z)$ into the set

$$\tilde{\Omega} = \mathbb{C} \setminus \{1\}$$

by defining

$$\tilde{f}(z) = \frac{1}{1-z} \ .$$

Example 2: Euler's gamma function is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt, \quad \operatorname{Re} z > 0 \ .$$

The formula defines an analytic function in the open right half plane. One can show that $\Gamma(z)$ can be continued analytically into the set

$$\mathbb{C} \setminus \{0, -1, -2, \ldots\} \ .$$

(This is not obvious.)

Theorem 3.3 *Let $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$ denote open connected sets. Suppose that f is analytic on Ω_1. Then there is be at most one analytic continuation of f into the set Ω_2.*

A difficulty is that there often is no natural choice for Ω_2.
Example 3:
Define

$$f(z) = \sum_{j=0}^{\infty} (-1)^j a_j z^j, \quad |z| < 1 \ ,$$

with

$$a_j = 2^{-j} 1 \cdot 3 \cdot 5 \cdot (2j-3) \ .$$

(Empty products are 1.) We have seen earlier: For $z = x, 0 \leq x < 1$,

$$f(x) = \sqrt{1+x} \ .$$

One can then show that

$$(f(z))^2 = 1 + z, \quad z \in B_1 \ .$$

3.6 Laurent series

An expression of the form

$$\sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

is called a Laurent series about z_0.

4 Asymptotic Evaluations of Integrals

In applications, one often encounters integrals that depend on a parameter. One wants to know how the integral 'behaves' as the parameter approaches zero or infinity.

4.1 A Regular Expansion Problem

Consider the following integral, depending on a parameter m:

$$I(m) = \int_0^{\pi/2} \frac{dt}{(1 - m\cos^2 t)^{1/2}}, \quad |m| << 1 \ .$$

Expand the function

$$f(\varepsilon) = (1 - \varepsilon)^{-1/2}$$

about $\varepsilon = 0$ using Taylor's formula. One obtains

$$(1 - \varepsilon)^{-1/2} = 1 + \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \dots$$

In this case, the Taylor series converges for $|\varepsilon| < 1$ towards $f(\varepsilon)$. If $g(t, m)$ denotes the integrand of the above integral $I(m)$, one obtains

$$g(t, m) = 1 + \frac{1}{2}\, m\cos^2 t + \frac{3}{8}\, m^2 \cos^4 t + \dots, \quad |m| < 1 \ .$$

We have

$$\int_0^{\pi/2} \cos^2 t \, dt = \frac{\pi}{4}$$

and

$$\int_0^{\pi/2} \cos^4 t \, dt = \frac{3\pi}{16} \ .$$

The latter integral can be obtained as follows:

$$\begin{aligned}
\cos^2 t &= \frac{1}{2}\,(1 + \cos 2t) \\
\cos^4 t &= \frac{1}{4}\,(1 + \cos 2t)^2 \\
&= \frac{1}{4}\,(1 + 2\cos 2t + \cos^2 2t) \\
&= \frac{1}{4} + \frac{1}{2}\cos 2t + \frac{1}{8}(1 + \cos 4t)
\end{aligned}$$

This yields that

$$\int_0^{\pi/2} \cos^4 t \, dt = \frac{\pi}{2}\left(\frac{1}{4} + \frac{1}{8}\right) = \frac{3\pi}{16} \ .$$

One obtains the following result:

$$I(m) = \frac{\pi}{2} + \frac{\pi}{8}m + \frac{9\pi}{8 \cdot 16}m^2 + \dots$$

In principle, one can expand to any order. In this case, the process leads to a convergent power series expansion of $I(m)$ valid for $|m| < 1$:

$$I(m) = \sum_{j=0}^{\infty} c_j m^j, \quad |m| < 1 \ .$$

The reason is that the power series

$$f(\varepsilon) = \sum_{j=0}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2j-1)}{2^j j!} \, \varepsilon^j$$

converges for $|\varepsilon| < 1$. We may substitute $\varepsilon = m \cos^2 t$ and then integrate over t.

4.2 A Problem Where the Interval Must Be Split

The following example is considerably more difficult. We consider

$$I(\varepsilon) = \int_0^{\pi/2} (\sin^2 t + \varepsilon \cos^2 t)^{-1/2} \, dt \quad \text{for} \quad 0 < \varepsilon \ll 1$$

and will determine the behaviour to leading order in ε.

We note two auxiliary results, used below for integration:

$$\frac{d}{dt} \ln(\tan \frac{t}{2}) = \frac{1}{\sin t}$$

and

$$\frac{d}{ds} \ln(s + \sqrt{1+s^2}) = \frac{1}{\sqrt{1+s^2}} \ .$$

To discuss the integral $I(\varepsilon)$, choose δ with

$$0 < \varepsilon << \delta^2 << 1 \ .$$

Write

$$I(\varepsilon) = \int_0^{\delta} + \int_{\delta}^{\pi/2} = I_1 + I_2 \ .$$

In I_2 we have

$$\varepsilon \cos^2 t << \sin^2 t \ .$$

(The reason is that $\sin t \geq 2t/\pi \geq 2\delta/\pi$ for $t \in I_1$.) Thus,

$$
\begin{aligned}
I_2 &\approx \int_\delta^{\pi/2} \frac{dt}{\sin t} \\
&= \ln(\tan\frac{t}{2})|_\delta^{\pi/2} \\
&= -\ln(\tan(\delta/2)) \\
&\approx \ln(2/\delta)
\end{aligned}
$$

In I_1 we have

$$\sin^2 t + \varepsilon \cos^2 t \approx t^2 + \varepsilon$$

This yields

$$
\begin{aligned}
I_1 &\approx \frac{1}{\sqrt{\varepsilon}} \int_0^\delta \frac{dt}{\sqrt{1+\frac{t^2}{\varepsilon}}} \\
&= \ln\Big(\frac{\delta}{\sqrt{\varepsilon}} + \sqrt{1+\frac{\delta^2}{\varepsilon}}\Big) \\
&\approx \ln\Big(\frac{2\delta}{\sqrt{\varepsilon}}\Big) \\
&= \ln 2 + \ln\delta + \frac{1}{2}\ln(1/\varepsilon)
\end{aligned}
$$

Thereofore,

$$
\begin{aligned}
I(\varepsilon) &= I_1 + I_2 \\
&\approx \ln 2 + \ln\delta + \frac{1}{2}\ln\frac{1}{\varepsilon} + \ln 2 - \ln\delta \\
&= \frac{1}{2}\ln\frac{16}{\varepsilon}
\end{aligned}
$$

Remark: Since the integral $I(\varepsilon)$ does not depend on δ, the final result should not depend on δ though the two pieces, I_1 and I_2, clearly depend on δ.

4.3 An Example for Watson's Lemma

We formulate Watson's Lemma below, but first consider an example.

Consider the following integral for $r \to \infty$:

$$I(r) = \int_0^\infty \frac{e^{-rt}}{1+t}\,dt \ .$$

Integration by parts yields

$$
\begin{aligned}
I(r) &= \int_0^\infty e^{-rt}(1+t)^{-1}\,dt \\
&= -\frac{1}{r}\,e^{-rt}(1+t)^{-1}\Big|_{t=0}^{t=\infty} - \frac{1}{r}\int_0^\infty e^{-rt}(1+t)^{-2}\,dt \\
&= \frac{1}{r} - \frac{1}{r}\int_0^\infty e^{-rt}(1+t)^{-2}\,dt
\end{aligned}
$$

The process can be continued and one obtains the asymptotic expansion

$$
I(r) \sim \sum_{j=0}^\infty (-1)^j \frac{j!}{r^{j+1}} \quad \text{as} \quad r \to \infty \; .
$$

This is only an asymptotic expansion since the series does not converge.

Let us obtain the same result in a different way. The second way will lead us to Watson's Lemma.

We will use the Γ–function,

$$
\Gamma(x) = \int_0^\infty s^{x-1} e^{-s}\,ds, \quad x > 0 \; .
$$

It is easy to check that $\Gamma(1) = 1$. Integration by parts leads to the functional equation for the Γ–function:

$$
\begin{aligned}
\Gamma(x+1) &= \int_0^\infty s^x e^{-s}\,ds \\
&= -s^x e^{-s}\Big|_{s=0}^{s=\infty} + \int_0^\infty x s^{x-1} e^{-s}\,ds \\
&= x\Gamma(x)
\end{aligned}
$$

The functional equation,

$$
\Gamma(x+1) = x\Gamma(x), \quad x > 0 \; ,
$$

together with $\Gamma(1) = 1$ yields

$$
\Gamma(j+1) = j!, \quad j = 0, 1, \ldots
$$

Let us return to the interal $I(r)$. We have

$$
(1+t)^{-1} = 1 - t + t^2 - t^3 + \ldots = \sum_{j=0}^\infty (-1)^j t^j \quad \text{for} \quad |t| < 1 \; .
$$

Therefore,

$$
\begin{aligned}
I(r) &= \int_0^{1/2} \ldots + \int_{1/2}^\infty \ldots \\
&= \sum_{j=0}^\infty (-1)^j \int_0^{1/2} t^j e^{-rt}\,dt + R_1(t)
\end{aligned}
$$

with

$$R_1(r) = \int_{1/2}^{\infty} e^{-rt}(1+t)^{-1}\, dt \ .$$

It is easy to see that

$$|R_1(r)| \leq \frac{1}{r}\, e^{-r/2} \ .$$

Thus, $R_1(r)$ decays exponentially to zero as $r \to \infty$. If one expands $I(r)$ in terms of powers of $1/r$, then $R_1(r)$ is negligable.

Also,

$$\int_0^{1/2} t^j e^{-rt}\, dt \sim \int_0^{\infty} t^j e^{-rt}$$

for large r. Finally,

$$\begin{aligned} \int_0^{\infty} t^j e^{-rt}\, dt &= r^{-j-1} \int_0^{\infty} s^j e^{-s}\, ds \\ &= \frac{j!}{r^{j+1}} \end{aligned}$$

One arrives at the same asymptotic expansion as above,

$$I(r) \sim \sum_{j=0}^{\infty} (-1)^j \frac{j!}{r^{j+1}} \quad \text{as} \quad r \to \infty \ .$$

4.4 Watson's Lemma

Consider an integral of the form

$$I(r) = \int_0^a t^{\lambda} f(t) e^{-rt}\, dt \ .$$

We want to know the asymptotic behavior of $I(r)$ as $r \to \infty$.

Assume

1. $$0 < a \leq \infty \ , \quad \lambda > -1$$
2. f is continuous and satisfies a bound $|f(t)| \leq Ae^{\alpha t}$.
3. f is analytic near $t = 0$:

$$f(t) = \sum_{j=0}^{\infty} a_j t^j, \quad |t| < t_0 \leq a \ .$$

Proceeding as in the above example, one obtain

$$I(r) \sim \frac{1}{r^{\lambda}} \sum_{j=0}^{\infty} a_j \frac{\Gamma(\lambda + j + 1)}{r^{j+1}} \quad \text{as} \quad r \to \infty \ .$$

4.5 A Highly Oscillatory Integral

For $r \gg 1$ the function

$$t \to e^{irt}$$

is highly oscillatory. Precisely, if r is a positive integer and t varies from 0 to 2π, then e^{irt} goes through r full oscillations. (If t is time, then r is called the frequency in e^{irt}. If t is length, then r is called the wave number.)

Integrals involving terms like e^{irt} for large real r are often called highly oscillory. A very simple example is:

$$\begin{aligned} I(r) &= \int_0^1 e^{irt} dt \\ &= \frac{\sin r}{r} + \frac{i}{r}(1 - \cos r) \end{aligned}$$

This can be obtained by integrating

$$e^{irt} = \cos(rt) + i \sin(rt)$$

or by noting that e^{irt} has the integral $\frac{1}{ir} e^{irt}$.

We note the following: Since $|e^{irt}| = 1$ for all real r and all real t one might expect that $I(r)$ is of order one for all r. However, one clearly obtains

$$I(r) = \frac{1}{ir} e^{irt} \Big|_{t=0}^{t=1} ,$$

thus $|I(r)| \le 2/r$. For large r, the high oscillations in e^{irt} lead to cancellations that reduce the integral.

4.6 More General Oscillatory Integrals

Consider (for finite a, b)

$$I(r) = \int_a^b A(t) e^{iru(t)} \, dt \ .$$

The function $A(t)$ is called the amplitude and the function $u(t)$, which is assumed to be real, is called the phase. One wants to understand the behavior of $I(r)$ for $r \to \infty$.

Assume

1. $A(t), u(t)$ are smooth; $u(t)$ is real.
2. $u'(t) \ne 0$ for all $a \le t \le b$

Obtain

$$\begin{aligned} I(r) &= \frac{1}{ir}\int_a^b \frac{A(t)}{u'(t)}\,\frac{d}{dt}e^{iru(t)}\,dt \\ &= \frac{1}{ir}\frac{A(t)}{u'(t)}\Big|_a^b - \frac{1}{ir}\int_a^b \Big(\frac{A(t)}{u'(t)}\Big)' e^{iru(t)}\,dt \end{aligned}$$

Note that the integral in the last line has the same form as the original integral, with $A(t)$ replaced by

$$A_1(t) = \Big(\frac{A(t)}{u'(t)}\Big)' \ .$$

Therefore, by repeating the process, one can derive an asymptotic expansion for $I(r)$.

4.7 The Method of Stationary Phase

Consider $I(r)$ as above but assume that there exists $a < t_0 < b$ with $u'(t_0) = 0, u'(t) \neq 0$ for $t \neq t_0$. Assume that $u''(t_0) \neq 0$ and $A(t_0) \neq 0$.

At $t = t_0$ the phase function $u(t)$ becomes stationary, i.e., to leading order in t, it does not change for t near t_0. If $u(t)$ does not change, then the function

$$e^{iru(t)}$$

does not oscillate. Therefore, for $t \sim t_0$, the cancellation effects are reduced when integrating $A(t)e^{iru(t)}$ near t_0. In other words, one expects the main contributions of $I(r)$ to come from $t \sim t_0$.

Choose a small $\delta > 0$ and split the interval $a \leq t \leq b$,

$$I(r) = \int_a^{t_0-\delta} + \int_{t_0+\delta}^{t_0-\delta} + \int_{t_0+\delta}^{b} \ .$$

We will see below that our final approximation for $I(r)$ does not depend on δ.

As shown above, the first and the last integral are $\mathcal{O}(\frac{1}{r})$. To approximate the middle integral, we use

$$A(t) \approx A(t_0), \quad u(t) \approx u(t_0) + \frac{u''(t_0)}{2}(t-t_0)^2$$

in $t_0 - \delta \leq t \leq t_0 + \delta$. Obtain

$$I(r) \approx A(t_0)e^{iru(t_0)}J(r,\delta)$$

with

$$J(r,\delta) = \int_{t_0-\delta}^{t_0+\delta} e^{i\alpha(t-t_0)^2}\,dt, \quad \alpha = \frac{1}{2}ru''(t_0) \ .$$

Assume, for definiteness, that $u''(t_0) > 0$. Note that $\alpha \to \infty$ as $r \to \infty$. Using the substitution

$$x = \sqrt{\alpha}(t - t_0)$$

we obtain

$$J(r,\delta) = \frac{1}{\sqrt{\alpha}} \int_{-\delta\sqrt{\alpha}}^{\delta\sqrt{\alpha}} e^{ix^2}\, dx \ , \quad \alpha = \frac{1}{2}\, ru''(t_0) \ .$$

We now evaluate

$$J(R) = \int_0^R e^{ix^2} dx \quad \text{for} \quad R \to \infty \ .$$

By Cauchy's theorem,

$$\begin{aligned} J(R) &= \int_{\mathcal{C}_{2R}} e^{iz^2} dz + \int_{\mathcal{C}_{3R}} e^{iz^2} dz \\ &=: J_2(R) + J_3(R) \end{aligned}$$

where $\mathcal{C}_{2R}$ has the parametrization

$$z(s) = (1+i)s, \quad 0 \le s \le R \ ,$$

and $-\mathcal{C}_{3R}$ has the parametrization

$$z(s) = R + is, \quad 0 \le s \le R \ .$$

We first estimate $J_3(R)$. Note that, on $\mathcal{C}_{3R}$:

$$z^2 = (R+is)^2 = R^2 - s^2 + 2Rsi$$

thus

$$|e^{iz^2}| = e^{-2Rs}$$

and

$$\begin{aligned} |J_3(R)| &\le \int_0^R e^{-2Rs}\, ds \\ &\le \frac{1}{2R} \end{aligned}$$

Since $R \sim \sqrt{r}$ the contribution to $J(r,\delta)$ coming from $J_3(R)$ if $\sim 1/r$.

For $J_2(R)$ we obtain

$$\begin{aligned} J_2(R) &= (1+i) \int_0^R e^{i(1+i)^2 s^2}\, ds \\ &= (1+i) \int_0^R e^{-2s^2}\, ds \\ &= \frac{1+i}{\sqrt{2}} \int_0^{\sqrt{2}R} e^{-\sigma^2}\, d\sigma \end{aligned}$$

Therefore,

$$J_2(R) \to \frac{1+i}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \quad \text{as} \quad R \to \infty \ .$$

Here

$$\frac{1+i}{\sqrt{2}} = e^{i\pi/4} \ .$$

Recall that

$$\alpha = \frac{1}{2} r u''(t_0), \quad R = \delta\sqrt{\alpha} \ .$$

One obtains that

$$\begin{aligned} J(r,\delta) &= \frac{2}{\sqrt{\alpha}} J(R) \\ &\approx \frac{2}{\sqrt{\alpha}} J_2(R) \\ &\approx \sqrt{\frac{2\pi}{r u''(t_0)}} \, e^{i\pi/4} \end{aligned}$$

Therefore,

$$\begin{aligned} I(r) &= \int_a^b A(t) e^{iru(t)} \, dt \\ &\approx A(t_0) e^{i(ru(t_0)+\frac{\pi}{4})} \sqrt{\frac{2\pi}{u''(t_0)}} \frac{1}{\sqrt{r}} \end{aligned}$$

This describes the leading order behavior of the integral as a product of a decaying factor, $1/\sqrt{r}$, an oscillatory factor, $e^{iru(t_0)}$, and a constant,

$$A(t_0) \, e^{i\pi/4} \sqrt{\frac{2\pi}{u''(t_0)}} \ .$$

If $u''(t_0) < 0$ then one must replace $u''(t_0)$ by $|u''(t_0)|$.

4.8 Remark on Fresnel Integrals

Using Cauchy's theorem, we have shown above that

$$\lim_{R\to\infty} \int_0^R e^{ix^2} \, dx = \frac{1+i}{4} \sqrt{2\pi} \ .$$

Taking real and imaginary parts, one obtains:

$$\lim_{R\to\infty} \int_0^R \cos(x^2) \, dx = \lim_{R\to\infty} \int_0^R \sin(x^2) \, dx = \frac{1}{4} \sqrt{2\pi} \ .$$

The above integrals, called Fresnel integrals, appear in applications to optics.

4.9 Stirling's Formula

Sterling gave the following asymptotic approximation for $r!$ as $r \to \infty$:

$$r! = \Gamma(r+1) \sim \sqrt{2\pi r}\, r^r e^{-r} \quad \text{as} \quad r \to \infty \ . \tag{4.1}$$

The advantage of the expression on the right side is that one can manipulate it better using the rules of calculus. For example, one can integrate it and differentiate it.

Derivation of Sterling's formula: We have

$$\begin{aligned} \Gamma(r+1) &= \int_0^\infty s^r e^{-s}\, ds \\ &= r^{r+1} \int_0^\infty t^r e^{-rt}\, dt \\ &=: r^{r+1} J(r) \end{aligned}$$

where we have used the substitution $s = rt$.

Since

$$t^r = e^{r \ln t}$$

we obtain:

$$J(r) = \int_0^\infty e^{r(\ln t - t)}\, dt \ .$$

The function

$$u(t) = \ln t - t, \quad t > 0 \ ,$$

satisfies

$$u'(t) = \frac{1}{t} - 1, \quad u''(t) = -\frac{1}{t^2} < 0, \quad t > 0 \ ,$$

which implies that $u(t)$ attains its maximum at $t_0 = 1$. Since

$$u(1) = -1, \quad u'(1) = 0, \quad u''(1) = -1 \ ,$$

we have

$$u(t) \sim -1 - \frac{1}{2}(t-1)^2 \quad \text{for} \quad t \sim 1 \ .$$

This yields:

$$
\begin{aligned}
J(r) &\sim \int_{1-\delta}^{1+\delta} \exp\Big(r(-1-\frac{1}{2}(t-1)^2)\Big)\,dt \\
&= e^{-r}\int_{1-\delta}^{1+\delta} \exp\Big(-\frac{r}{2}(t-1)^2)\Big)\,dt \\
&= e^{-r}\int_{-\delta}^{\delta} \exp\Big(-\frac{r}{2}x^2)\Big)\,dx \\
&= e^{-r}\sqrt{2/r}\int_{-\delta\sqrt{r/2}}^{\delta\sqrt{r/2}} e^{-y^2}\,dy
\end{aligned}
$$

In the last equation we have used the substitution

$$x\sqrt{r/2} = y\ .$$

As $r \to \infty$, the final integral tends to $\sqrt{\pi}$. Therefore,

$$J(r) \sim e^{-r}\sqrt{2\pi/r}$$

and

$$r! = \Gamma(r+1) \sim r^{r+1}e^{-r}\,\sqrt{2\pi/r} = \sqrt{2\pi r}\,r^r e^{-r}\ .$$

4.10 The Method of Steepest Descent

Consider an integral of the form

$$I(r) = \int_A^B g(z)e^{rf(z)}\,dz \quad \text{as} \quad r \to \infty\ .$$

Example 1:

$$I(r) = \int_1^3 e^{irz^2} dz$$

Here

$$
\begin{aligned}
f(z) &= i(x+iy)^2 \\
&= -2xy + i(x^2-y^2)
\end{aligned}
$$

thus

$$u = -2xy, \quad v = x^2 - y^2$$

Consider the family of lines

$$u(x,y) = c \quad \text{and} \quad v(x,y) = c\ .$$

The first family are the hyperbolae

$$y = -\frac{c}{2x}$$

and the second family are the orthogonal hyperbolae

$$y = \pm\sqrt{x^2 - c} \; .$$

(The orthogonality relation follows generally from the Cauchy–Riemann equations.)

Choose the following path from $A = 1$ to $B = 3$ in the complex plane: $\mathcal{C}_1$ starts at $A = 1$ and follows the line $v = 1$ in decreasing u direction, from $u = 0$ to $u = -U$. Here $U > 0$ is a chosen number. The value of U will not be important.

$\mathcal{C}_2$ is determined by: $u = -U$ and v increases from $v = 1$ to $v = 9$. $\mathcal{C}_3$ is determined by: $v = 9$ and u increases from $-U$ to 0. Thus $\mathcal{C}_3$ ends at $B = 3$.

Then

$$I(r) = \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} \; .$$

The integral along $\mathcal{C}_2$ is exponentially small since $u = -U$ along $\mathcal{C}_2$,

$$\int_{\mathcal{C}_2} \sim e^{-rU} \quad \text{as} \quad r \to \infty \; .$$

The main contributions to $I(r)$ come from the integrals along $\mathcal{C}_1$ and $\mathcal{C}_3$ near the end points $A = 1$ and $B = 3$ where $u = 0$. Note that $v = 1$ along $\mathcal{C}_1$ and $v = 9$ along $\mathcal{C}_3$. Since v does not change along the curves, the oscillatory part can be taken out of the integral.

Approximation of $\int_{\mathcal{C}_1}$. It is important to choose a convenient parametrization of $\mathcal{C}_1$. Basically, we parametrize $\mathcal{C}_1$ by the u–values along $\mathcal{C}_1$. Note that, along $\mathcal{C}_1$,

$$\begin{aligned} f(z) &= iz^2 \\ &= -2xy + i(x^2 - y^2) \\ &= u + i \end{aligned}$$

where $0 \geq u \geq -U$. Then $\mathcal{C}_1$ has the parametrization $z(t)$ with

$$iz^2(t) = -t + i, \quad 0 \leq t \leq U \; .$$

This yields

$$z(t) = \sqrt{1 + it} \; .$$

(Note that $z(0) = 1$, thus the root lies in the first quadrant.)

Therefore,

$$dz = \frac{i}{2}(1 + it)^{-1/2} dt \; .$$

Obtain

$$\begin{aligned}
\int_{\mathcal{C}_1} e^{iz^2 r} dz &= \frac{i}{2}\int_0^U e^{(-t+i)r}(1+it)^{-1/2}\,dt \\
&= \frac{i}{2}\,e^{ir}\int_0^U e^{-tr}(1+it)^{-1/2}\,dt
\end{aligned}$$

To approximate the integral, we use Watson's lemma. We have

$$(1+it)^{-1/2} = 1 - \frac{i}{2}t + \mathcal{O}(t^2)\ .$$

This yields

$$\int_{\mathcal{C}_1} \ldots = \frac{i}{2}\,e^{ir}\Big(\frac{1}{r} - \frac{i}{2}\frac{1}{r^2} + \mathcal{O}(r^{-3})\Big)\ .$$

Approximation of $\int_{-\mathcal{C}_3}$. Parametrize $-\mathcal{C}_3$ by $z(t)$ where

$$iz^2(t) = -t + 9i, \quad 0 \le t \le U\ .$$

Obtain

$$\begin{aligned}
z(t) &= 3\Big(1 + \frac{it}{9}\Big)^{1/2} \\
z'(t) &= \frac{3}{2}\frac{i}{9}\Big(1 + \frac{it}{9}\Big)^{-1/2} \\
&= \frac{i}{6}\Big(1 + \frac{it}{9}\Big)^{-1/2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{-\mathcal{C}_3} e^{iz^2 r} dz &= \frac{i}{6}\int_0^U e^{(-t+9i)r}\Big(1+\frac{it}{9}\Big)^{-1/2}\,dt \\
&= \frac{i}{6}e^{9ir}\int_0^U e^{-tr}\Big(1 - \frac{it}{18} + \mathcal{O}(t^2)\Big)^{-1/2}\,dt \\
&= \frac{i}{6}e^{9ir}\Big(\frac{1}{r} - \frac{i}{18}\frac{1}{r^2} + \mathcal{O}(r^{-3})\Big)
\end{aligned}$$

Together,

$$I(r) = \Big(\frac{i}{2}e^{ir} - \frac{i}{6}e^{9ir}\Big)\frac{1}{r} + \Big(\frac{1}{4}e^{ir} - \frac{1}{6\cdot 18}e^{9ir}\Big)\frac{1}{r^2} + \mathcal{O}(r^{-3}) \quad \text{as} \quad r \to \infty\ .$$

Example 2:

$$I(r) = \int_A^B z^2 e^{ir(z^3+3z)}\,dz$$

with

$$A = -1 + i, \quad B = 1 + i \ .$$

Here

$$\begin{array}{rcl} u & = & -3x^2y + y^3 - 3y \\ v & = & x^3 + 3x - 3xy^2 \end{array}$$

The main contribution to $I(r)$ comes from $\int_{\mathcal{C}_2}$ where $\mathcal{C}_2$ is parametrized by $z(t)$ with

$$iz^3(t) + 3iz(t) = u(t) + iv = -2 - t^2, \quad v = 0$$

and

$$-\sqrt{3} \leq t \leq \sqrt{3} \ .$$

Obtain

$$\int_{\mathcal{C}_2} z^2 e^{f(z)r} dz = \int_{-\sqrt{3}}^{\sqrt{3}} z^2(t) e^{(-2-t^2)r} \frac{dz}{dt}(t)\, dt \ .$$

The main contribution comes from the integrand near $t = 0$. We have $z(0) = i, z^2(0) = -1$. One must obtain $\frac{dz}{dt}(0)$. Note that

$$f(i + dz) = -2 - 3(dz)^2 = -2 - (dt)^2 \ ,$$

thus

$$dz = \frac{1}{\sqrt{3}}\, dt$$

at $t = 0$. This yields

$$\begin{array}{rcl} \int_{\mathcal{C}_2} \dots & \sim & -\frac{1}{\sqrt{3}} e^{-2r} \int_{-\varepsilon}^{\varepsilon} e^{-t^2 r}\, dt \\ & \sim & -\frac{\pi}{\sqrt{3}} e^{-2r} \frac{1}{\sqrt{r}} \ . \end{array}$$