LAPLACE TRANSFORMS

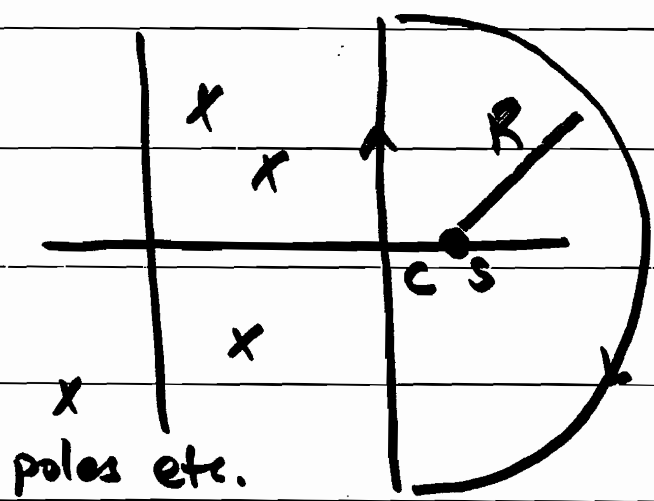
$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

If $\exists\, M$: $f(t)\, e^{-Mt} \to 0$ as $t \to \infty$,

then $F(s)$ analytic for $\operatorname{Re} s < M$.

Inversion:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, e^{st}\, ds$$



(or $c > M$)

Now: $\displaystyle \frac{1}{2\pi i} \int_0^\infty e^{-st}\, dt \int_{c-i\infty}^{c+i\infty} F(z)\, e^{zt}\, dz \quad (= F(s))$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z)\, dz \int_0^\infty e^{-(s-z)t}\, dt \quad , s > c$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F(z)}{s-z}\, dz$$

$$\frac{1}{2\pi i}\int_L \frac{F(z)}{s-z}\,dz = F(s) - \frac{1}{2\pi i}\int_{C_R}\frac{F(z)}{s-z}\,dz$$

if $f(t)e^{-Mt} \xrightarrow{t\to\infty} 0 \Rightarrow F(z) \to 0$ as $z \to \infty$ in $\operatorname{Re} z > c$.

Then $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{F(z)}{s-z} = F(s)$; but this is Cauchy's formul

Laplace integral transform pairs :
$$\begin{cases} f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds \\ F(s) = \int_0^\infty e^{-st}f(t)\,dt \end{cases}$$

Eg.

(i) $f(t) = 1$; $F(s) = \frac{1}{s}$, $\operatorname{Re} s > 0$

(ii) $f(t) = t^m$; $F(s) = \frac{m!}{s^{m+1}}$, $\operatorname{Re} s > 0$

(iii) $f(t) = e^{at}$; $F(s) = \frac{1}{s-a}$, $\operatorname{Re} s > a$

(iv) $f(t) = \begin{matrix}\cos at \\ \sin at\end{matrix}$ $F(s) = \frac{a}{s^2+a^2}$ $= \frac{s}{s^2+a^2}$, $\operatorname{Re} s > 0$

Restrictions on s define regions where $\int_0^\infty e^{-st} f(t)\,dt$ converges. But expressions for $F(s)$ define functions for all s. Here, analytic continuation gives $F(s)$ for all s.

Some properties: $\mathcal{L}\{f(t)\} = F(s)$

(i) $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$: shifting property

(ii) $$\mathcal{L}\left\{\frac{df}{dt}\right\} = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty f\, e^{-st}\,dt = sF(s) - f(0)$$

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s\,\mathcal{L}\left\{\frac{df}{dt}\right\} - f'(0) = s^2 F(s) - s f(0) - f'(0)$$

(iii) $$\mathcal{L}\{-t f(t)\} = \frac{dF}{ds} \qquad \left(\frac{dF}{ds} = -\int_0^\infty t f(t)\,dt\, e^{-st} \text{ etc.} \right)$$

$$\mathcal{L}\{t^2 f(t)\} = \frac{d^2 F}{ds^2}$$

(iv) Convolution: $H(s) = G(s)\,F(s) \longrightarrow h(t) = \int_0^t g(\tau) f(t-\tau)\,d\tau = g * f$

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} F(s) \qquad \left(\int_0^t f\,dt = 1 * f \right)$$

Ex. $y' + ay = \sin\beta x \qquad y(0) = 0$

$$\rightarrow sY + aY = \frac{\beta}{s^2+\beta^2}$$

$$Y(s) = \frac{\beta}{(s^2+\beta^2)(s+a)} = \frac{\beta^2}{a^2+\beta^2}\left(\frac{1}{s+a} - \frac{s}{s^2+\beta^2} + \frac{a}{s^2+\beta$$

$$\Rightarrow y(x) = \frac{\beta}{a^2+\beta^2}\left(e^{-ax} - \cos\beta x + \frac{a}{\beta}\sin\beta x\right)$$

Ex. $\left.\begin{array}{l}\ddot{x} + a^2\dot{y} = b \\ \ddot{y} - a^2\dot{x} = 0\end{array}\right\} \Rightarrow \left.\begin{array}{l} s^2X + a^2sY = \frac{b}{s} \\ s^2Y - a^2sX = 0\end{array}\right\}$ $\qquad X = \frac{b}{s(a^4+s^2)}$

$x_0 = \dot{x}_0 = y_0 = \dot{y}_0 = 0$ $\qquad Y = \frac{ba^2}{s^2(a^4+s^2)}$

Partial fractions: typical forms

$\frac{1}{s-a} \longrightarrow e^{at} \qquad \frac{1}{s^2+2\alpha s+w_0^2} \longrightarrow \frac{e^{-at}}{w}\sin wt \;;\; w = \sqrt{w_0^2-a^2}$

$\frac{1}{(s-a)^2} \longrightarrow te^{at} \qquad \frac{s-x}{s^2+2as+w_0^2} \longrightarrow e^{-at}\cos wt$

Here: $X = \frac{b}{s(a^4+s^2)} = \frac{b}{a^4 s} - \frac{b}{a^4}\frac{s}{a^4+s^2}$

$Y = \frac{b}{s^2(a^4+s^2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{a^4+s^2}$

$= \frac{b}{a^4 s^2} - \frac{b}{a^4} \cdot \frac{1}{a^4+s^2}$

Hence:

$x = \frac{b}{a^4} - \frac{b\cos a^2 t}{a^4}$; $y = \frac{bt}{a^2} - \frac{b}{a^4}\sin a^2 t$

Dimension Check: $a^2 \sim \frac{1}{\text{time}}$, $b \sim \frac{\text{distance}}{\text{time}^2}$

consistent with $b/a^4 \sim$ distance, $\frac{b}{a^2} \sim$ velocity

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In general: system of ODE $\longrightarrow$ system of algebraic equ.

$A\frac{d\underline{y}}{dt} + B\underline{y} = \underline{c}$ $\longrightarrow$ $(sA+B)Y = \underline{c} + A\underline{k}$

$\underline{y}(0) = \underline{k}$

$\hookrightarrow$ invert (if nonsingular)

Get rational functions of s

$\rightarrow$ partial fractions.

Example (the error function)

$$\left.\begin{aligned} y'' + xy' + y &= 0 \\ y(0) = 1, \ y'(0) &= 0 \end{aligned}\right\}$$

$\mathcal{L}$: $\quad s^2 Y - s y_0 - y_0' - \frac{d}{ds}(sY) + Y = 0$

$$\Rightarrow \frac{dY}{ds} - sY = -1 \qquad (*)$$

to solve, need some b.c. If $|y| < me^{kx}$

then $\cancel{Ky}$ $|Y| \le m\int_0^\infty e^{(k-s)x}\,dx = \frac{1}{s-k} \to 0$ as $s \to \infty$.

$\therefore$ Require $Y(s) \to 0$ as $\mathrm{Re}\, s \to \infty$

Solving $(*)$: $\quad Y(s) = -e^{+s^2/2} \int_\infty^s e^{-\sigma^2/2}\,d\sigma$ or

$$Y(s) = \sqrt{2}\, e^{s^2/2} \int_{s/\sqrt{2}}^\infty e^{-u^2}\,du$$

"Complementary rror function"

Error function: $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2} d\zeta$

$(\operatorname{erf}(0) = 0,\ \operatorname{erf}(\infty) = 1)$

Comp. erf: $\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\zeta^2} d\zeta$

So $\quad Y(s) = \sqrt{2\pi}\, e^{s^2/2} \operatorname{erfc}(s/\sqrt{2})$

Can find inverse in tables. But consider

a trick: $\quad Y(s) = \int_s^\infty e^{(s^2-\sigma^2)/2} d\sigma \quad$; let $\sigma = s + t$

$s^2 - \sigma^2 = -t(2\sigma + t)$ $\quad \Rightarrow Y(s) = \int_0^\infty e^{-st} e^{-t^2/2} dt = \mathcal{L}\{e^{-t^2/2}\}$

$\Rightarrow y(t) = e^{-t^2/2}$

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Aside: erfc(x) for $x \gg 1$.
(1)

$$\int_x^\infty e^{-\zeta^2}\,d\zeta = -\frac{e^{-x^2}}{2x} + \frac{1}{2}\int_x^\infty \frac{e^{-\zeta^2}}{\zeta}\,d\zeta$$

Now $\displaystyle\int_x^\infty \frac{e^{-\zeta^2}}{\zeta}\,d\zeta < \int_x^\infty \frac{\zeta}{x^2} e^{-\zeta^2}\,d\zeta = \frac{1}{(2x)^2}\, e^{-x^2}$

i.e. $\displaystyle\int_x^\infty e^{-\zeta^2}\,d\zeta = \frac{e^{-x^2}}{2x} + \text{error}$, $\displaystyle|\varepsilon_1| < \frac{e^{-x^2}}{(2x)^2}$

Can refine this process by repeatedly integrating by parts: after m integrations:

$$\int_x^\infty e^{-\zeta^2}\,d\zeta = e^{-x^2}\left\{ \frac{0!}{2x} + \frac{1!}{(2x)^2} + \frac{2!}{(2x)^3} + \dots + \frac{m!}{(2x)^{m+1}} + \text{error} \right\}$$

as before, can show: $\displaystyle|\text{error}| < \frac{(m+1)!}{(2x)^{m+2}}$

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A small detail (!): Series diverges for all x!

However: if we fix m (the number of terms), then error becomes small as $x \to \infty$. **Asymptotic Expansion**

Aside (2): Differential-Difference equations

$$y' = y(t-1)$$

$$\mathcal{L}\{y(t-1)\} = \int_0^\infty y(t-1)e^{-st}dt = e^{-s}\int_{-1}^\infty y(t)e^{-st}dt$$

$$= e^{-s}\int_{-1}^0 y(t)e^{-st}dt + e^{-s}Y(s)$$

$\hookrightarrow$ initial data now needed over $-1 \le t \le 0$.

Say $y(t) = 1$, $-1 < t < 0$: then

$$\mathcal{L}\{y(t-1)\} = \frac{1}{s} - \frac{e^{-s}}{s} + e^{-s}Y(s)$$

then: $sY - 1 = \frac{1}{s} - \frac{e^{-s}}{s} + e^{-s}Y$

$$\Rightarrow Y(s) = \frac{1}{s} - \frac{1}{s(s-e^{-s})}$$

$$\Rightarrow y(t) = 1 + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(s-e^{-s})}\,ds$$

Impossible to invert in closed form.

Q: how does $y(t)$ behave for $t \to \infty$?

Poles: zeros of $s(s-e^{-s})$

$Re\, s \to \infty$, $s(s-e^{-s}) \to \infty$: All zeros in left half plane $\underline{Re\, s <}$

Numerically: pole at $s = .567$; all others $Re\, s < 0$.

Now $y(t) = 1 + \sum_j \frac{e^{s_j t}}{s_j(1+s_j)}$

(residue at simple pole $s = s_j$:

$$\lim_{s \to s_j} (s - s_j) \left. \frac{e^{st}}{s(s - e^{-s})} \right| = \frac{e^{s_j t}}{s_j (1 + s_j)}$$

$$e^{-s_j} = s_j$$

(l'Hospital $\quad \frac{s - s_j}{s - e^{s}} \longrightarrow \frac{1}{1 + e^{-s}} = \frac{1}{1 + s_j}$

All terms decay except pole at .567, i.e.

$$y(t) \sim \frac{e^{.567 t}}{.567(1 + .567)} \approx 1.125\, e^{.567 t}, \quad t \to \infty$$

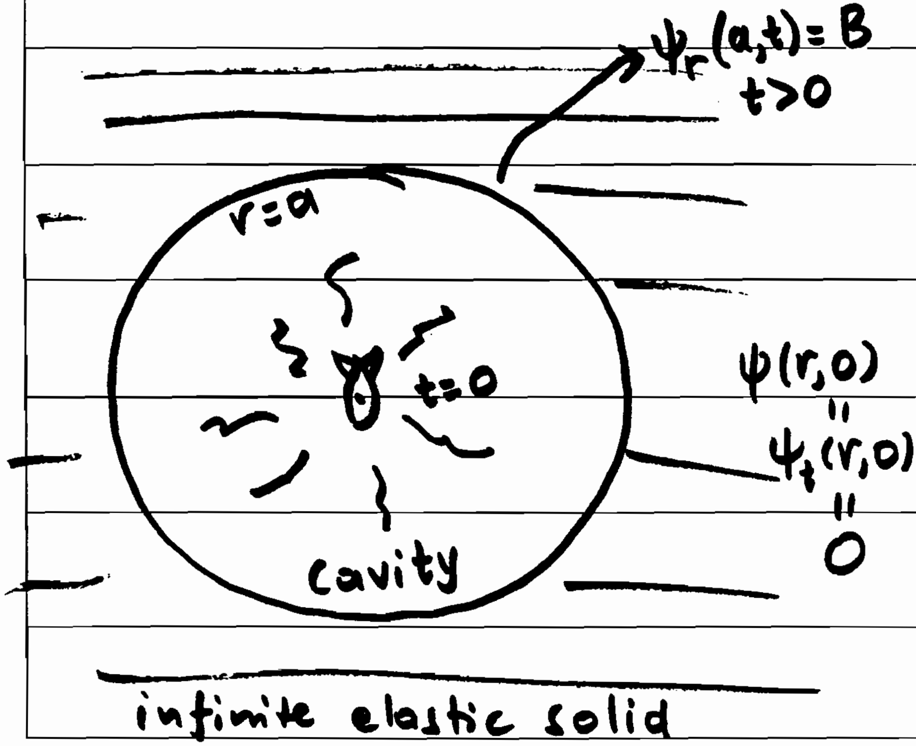
In general, we need to approximate $f(t)$ when inversion not possible explicitly; easiest cases

$t \sim 0, \quad t \sim \infty.$

Laplace transforms and P.D.E.

Example: the propagation of spherically symmetric waves in an infinite elastic solid is described by

$$\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\psi)$$



where $\underline{c}$ is the wave speed and $\underline{\psi}$ is a potential function related to the radial displacement $\underline{u}$ of the material by

$$u = \frac{\partial \psi}{\partial r}.$$

An elastic medium such as this has a spherical cavity of radius $\underline{a}$ and is at rest at time $t=0$; that is

$$\psi = \frac{\partial \psi}{\partial t} = 0 \quad \text{at } t=0.$$

At time $t=0^+$ a bomb is exploded in the cavity so that the walls are pushed out by a uniform radial displacement B; that is

$$u(a,t) = B\,H(t) = \begin{cases} B, & t>0 \\ 0 & t<0 \end{cases}.$$

We can use Laplace transforms (in t) to find the solution for the displacement $u(r,t)$, $r>a$, $t>0$.

Collecting: $\partial_r^2(\phi) = \frac{1}{c^2}\partial_t^2(\phi)$

let $r\psi = \phi$ $\qquad \phi(r,0) = \phi_t(r,0) = 0$

$$\psi_r\Big|_{r=a} = BH(t)$$

Let $\Phi = \mathcal{L}\{\phi\}$, $\Psi = \mathcal{L}\{\psi\}$. Then

$$\partial_r^2\Phi - \frac{s^2}{r^2}\Phi = 0 \Rightarrow \Phi(r) = A(s)\,e^{-\frac{sr}{c}}$$

(suppress growing exponential: medium is at rest at $r=\infty$).

$$\Psi_r = \partial_r\left(\frac{\Phi}{r}\right) = \left(-\frac{1}{r^2} - \frac{s}{rc}\right)Ae^{-sr/c}\Big|_{r=a} = \frac{B}{s}$$

$$A(s) = -\frac{Ba^2c}{s(c+as)}\,e^{sa/c}$$

$$\Psi_r(r,s) = \frac{Ba^2(c+rs)}{r^2 s(c+as)}\,e^{-\frac{s}{c}(r-a)}$$

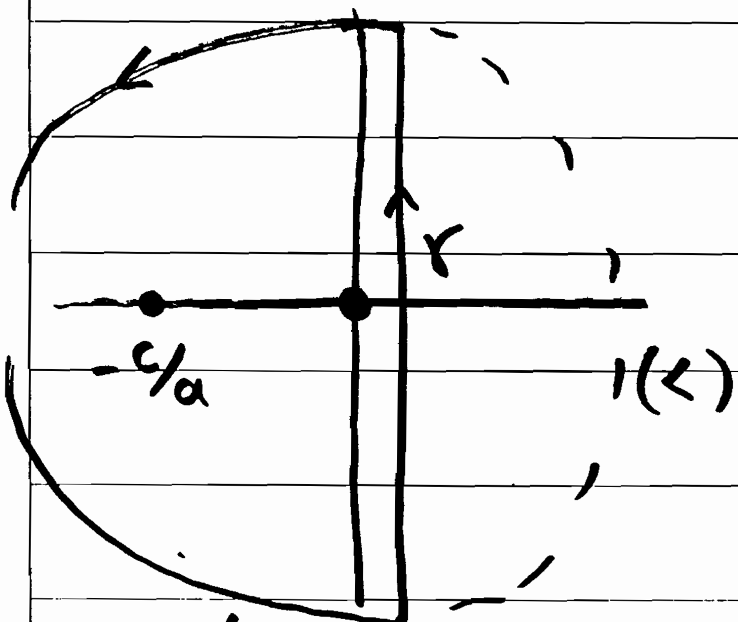
so $u(r,t) = \psi_r(r,t) =$

$$= \frac{Ba^2}{r^2} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{c+rs}{s(c+as)}\, e^{s\left[t-\frac{1}{c}(r-a)\right]}\, ds$$

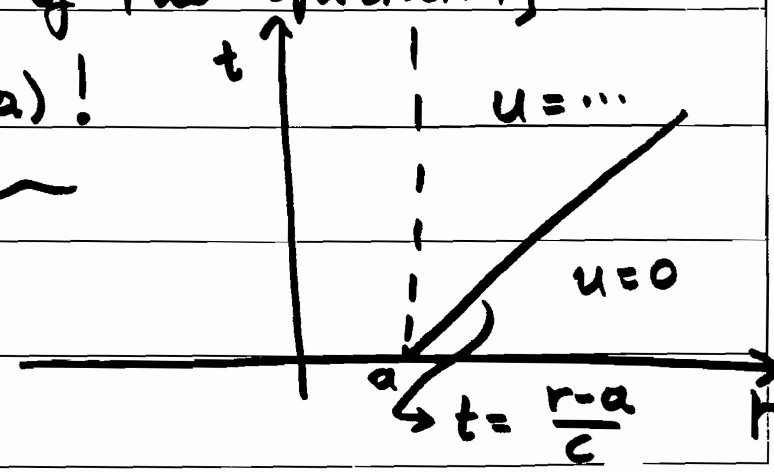
$$= \frac{Ba^2}{r^2} \sum (\text{res})$$

But we must be careful! How contour is completed depends on the sign of the quantity

$$\underbrace{t - \frac{1}{c}(r-a)}\ !$$



contour for $t - \frac{1}{c}(r-a) > 0$



* for $t - \frac{1}{c}(r-a) < 0$ (wave from explosion has not yet reached location), contour is completed to the right, and

$$u(r,t) = 0, \quad t < \frac{r-a}{c}.$$

* for $t - \frac{r-a}{c} > 0$, contour is completed to left:

$$u(r,t) = \begin{cases} 0, & t < (r-a)/c \\ \frac{Ba^2}{r^2}\left\{ res(s=0) + res(s=-c/a) \right. \end{cases}$$

$$\frac{c+rs}{c+as} e^{s[\dots]}\Big|_{s=0} = 1$$

$$\frac{c+rs}{s} e^{s[\dots]}\Big|_{s=-c/a} = (r-a)\, e^{-\frac{c}{a}\left[t - \frac{r-a}{c}\right]}$$

Example (a harder contour integral)

$$u_{xx} = \frac{1}{c^2} u_{tt} \quad , \quad 0 < x < \ell, \quad t > 0$$

$$u(0,t) = 0 \quad , \quad u(\ell,t) = k \quad (\text{const})$$

$$u(x,0) = u_t(x,0) = 0$$

Let $V = \mathcal{L}\{u\}$:

$$V_{xx} - \frac{s^2}{c^2} V = 0$$

$$V(0,s) = 0, \quad V(\ell,s) = \frac{k}{s}$$

$$V(x,s) = A(s)\, e^{s/c\, x} + B(s)\, e^{-s/c\, x}$$

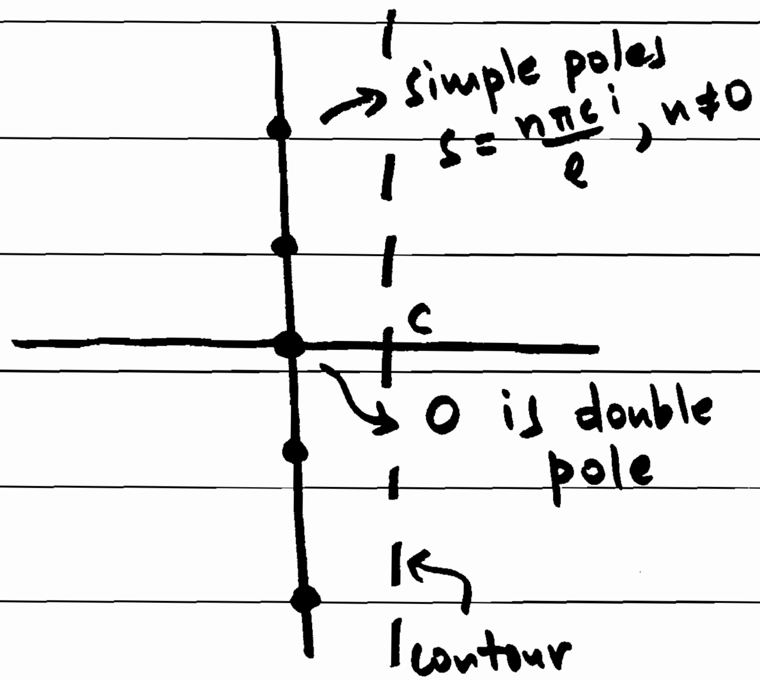
$$V(0,s) = A + B = 0$$

$$V(\ell,s) = A\left(e^{s\ell/x} - e^{-s\ell/x}\right) = \frac{k}{s}$$

$$V(x,s) = \frac{k}{s} \frac{\sinh(sx/c)}{\sinh(s\ell/c)}$$

$$u(x,t) = \frac{k}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s}\,\frac{\left(e^{sx/c} - e^{-sx/c}\right)}{\left(e^{s\ell/c} - e^{-s\ell/c}\right)}\,ds$$

$$e^{2s\ell/c} = 1 \Rightarrow \frac{2s\ell}{c} = 2n\pi i \Rightarrow s_n = \frac{n\pi c i}{\ell}$$



The two terms in numerator:

$$e^{s(t+x/c)}, \quad e^{s(t-x/c)}$$

represent left- and right-traveling waves.

Inversion must acount for

$t + x/c \gtrless 0$ (always >0)

$t - x/c \gtrless 0$ (can be either)

(will do this later).