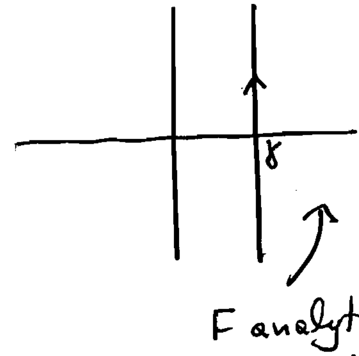
Laplace transforms (continued)

1
$$\begin{cases} F(s) = \int_0^t f(t)\, e^{-st}\, dt \\ f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\, e^{st}\, ds \end{cases}$$

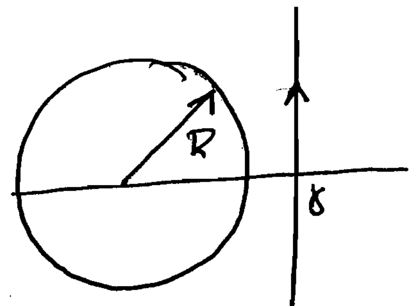


In general, inversion cannot be done exactly.
The two limits, $t \to 0^+$, $t \to \infty$ readily lend themselves to analysis

(i) t small, $\Rightarrow$ inversion depends on large s behavior of F.

Say, for $R = |s| > R_0$

$$F(s) = \sum_1^\infty \frac{a_m}{s^m} \quad \text{convergent}$$



By uniform convergence:

$$\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \left(\sum_1^\infty \frac{a_m}{s^m} \right) ds = \sum_1^\infty a_m \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s^m}\, ds = \sum_1^\infty \frac{a_m t^{m-1}}{(m-1)!}$$

If $F(s) \sim \sum_{m=1}^\infty \frac{a_m}{s^{\nu_m}}$; $\mathrm{Re}(s) > M_0$ (convergent or asymptotic) then

we can still interchange summation and integration:

$$f(t) \sim \sum_1^\infty a_m \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s^{\nu_m}}\, ds \approx \sum_1^\infty \frac{a_m t^{\nu_m - 1}}{\Gamma(\nu_m)}$$

(t small for convergence)

(2) Calculation: the transform pair

$$f(t) = t^{\alpha} \; ; \quad F(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\therefore \int_0^{\infty} t^{\alpha} e^{-st} dt = \frac{1}{s^{\alpha+1}} \int_0^{\infty} (st)^{\alpha} e^{-st} d(st) = \frac{1}{s^{\alpha+1}} \underbrace{\int_0^{\infty} u^{\alpha} e^{-u} du}_{\equiv\, \Gamma(\alpha+1)}$$

The Gamma function:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \; ; \quad Re(\alpha) > 0 \; \left\{ \begin{array}{l} \text{for convergence} \\ \text{at } x=0 \end{array} \right.$$

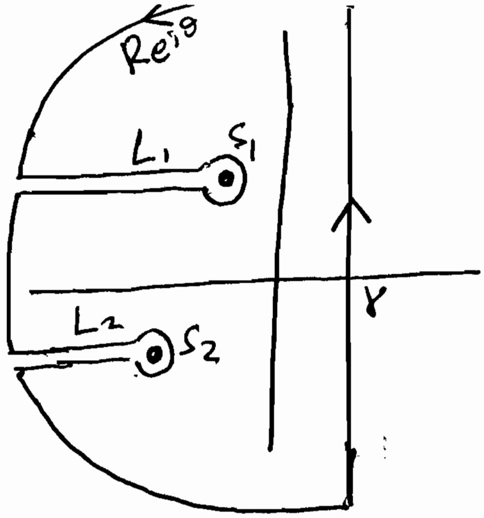
Note: $\Gamma(\frac{1}{2}) = \int_0^{\infty} x^{-1/2} e^{-x} dx = 2\int_0^{\infty} e^{-u^2} du = \sqrt{\pi}$ (error integral)

$$= 2 erf(\infty)$$

$\alpha = n+1$, integer : $\Gamma(n+1) = \int_0^{\infty} x^{n} e^{-x} dx = n!$

Aside

(ii) The limit $t \to \infty$ (this subject is usually treated by methods of asymptotics for integrals). In general singularities of $F(s)$ determine behavior of $f(t)$ as $t \to \infty$. The singularity with the largest real part gives dominant behavior. Let $F(s)$ have two singularities, with $Re(s_1) > Re(s_2)$



$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \simeq \underset{\uparrow}{A_1 e^{s_1 t}} + A_2 e^{s_2 t} + \dots$$

dominant term

Eg. if s_1 pole:

$$\int_{L_1} \sim e^{s_1 t} \operatorname*{res}_{s=s_1} F(s)$$

If s_1 branch point: $F(s) = k(s-s_1)^{a-1} +$ smaller terms

$$f(t) \sim \frac{e^{s_1 t}}{2\pi i} k \int_{L_1} \frac{e^{(s-s_1)t}}{(s-s_1)^{1-a}}\, ds \approx \frac{k}{\pi} e^{s_1 t} t^{-a}\,\Gamma(a)\sin\pi a$$

Define $\quad z! := \Gamma(z+1) = \int_0^\infty x^z e^{-x}\,dx \; ; \; \mathrm{Re}(z) > -1$

Properties

(*) recurrence relation $\quad \Gamma(z+1) = z\,\Gamma(z)$

$$\Gamma(z+1) = \int_0^\infty x^z e^{-x}\,dx = -\int_0^\infty x^z\, d(e^{-x})$$

$$= \underbrace{-x^z e^{-x}\Big|_0^\infty}_{=0} + z\int_0^\infty x^{z-1}e^{-x}\,dx = z\,\Gamma(z)$$

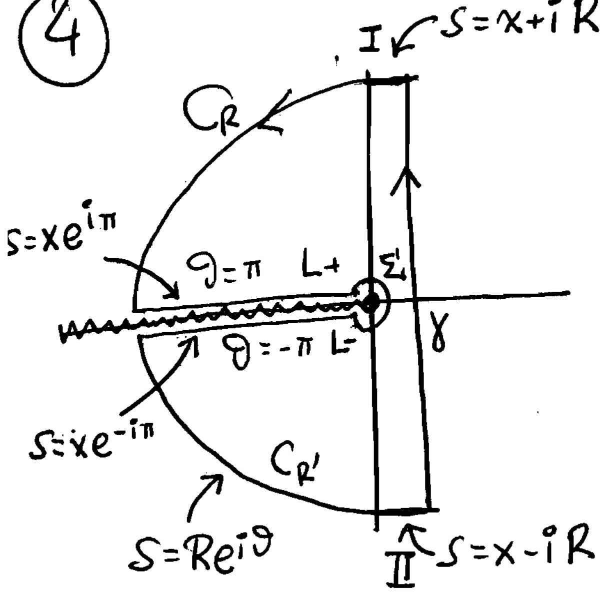
* Can use recurrence to analytically continue $\Gamma(z)$ to $\mathrm{Re}\, z < -1$

* $\Gamma(z)$ is analytic everywhere except for poles at $z = 0, -1, -2, \dots$

* $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin\pi\alpha}$ (homework)

The inverse: verify that the inversion formula gives

$$\mathcal{L}^{-1}\left\{\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right\} = t^\alpha, \quad \mathrm{Re}\,\alpha > -1.$$



We work with α real, and use analytic continuation.

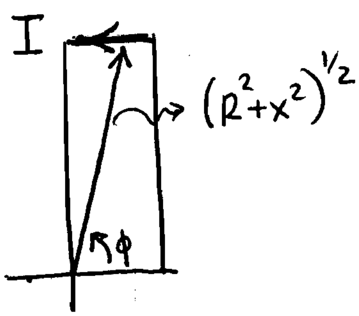
$$f(t) \overset{?}{=} \frac{\Gamma(\alpha+1)}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{s^{\alpha+1}}\,ds$$

Now $\oint_C \frac{e^{st}}{s^{\alpha+1}}\,ds = 0 \Rightarrow$

$$\Rightarrow \int_\gamma = -\int_{C_R} - \int_{C_R'} - \int_{I} - \int_{II} - \int_{\Sigma} - \int_{L^+} - \int_{L^-}$$

(1) At I : $s = x+iR$

$$\left|\frac{e^{st}ds}{s^{\alpha+1}}\right| = \left|\frac{e^{xt}e^{iRt}dx}{(R^2+x^2)^{\frac{\alpha+1}{2}}e^{i\phi(\alpha+1)}}\right| = \frac{e^{xt}dx}{(R^2+x^2)^{\alpha+1/2}} \xrightarrow{R\to\infty} 0 \quad \text{for } \alpha>-1$$



Similarly for II : $\int_I + \int_{II} \xrightarrow{R\to\infty} 0.$

(2) At C_R, $s = Re^{i\theta}$, $\frac{\pi}{2} \le \theta \le \pi$

$$\left|\int \frac{e^{st}ds}{s^{\alpha+1}}\right| \le \int_{\pi/2}^{\pi}\left|\frac{e^{tR\cos\theta}e^{itR\sin\theta}Rie^{i\theta}d\theta}{R^{\alpha+1}e^{i\theta(\alpha+1)}}\right| \le \int_{\pi/2}^{\pi}\frac{e^{tR\cos\theta}}{R^{\alpha}}d\theta$$

(Jordan lemma) $\quad = R^{-\alpha}\int_0^{\pi/2} e^{tR\sin\theta}d\theta \le R^{-\alpha}\frac{1}{R}(1-e^{-R}) \longrightarrow 0$ for $\alpha > -1$

Similarly for C_R' : $\int_{C_R} + \int_{C_R'} \xrightarrow{R\to 0} 0$

(3) Finally, the integral at Σ: $s = re^{i\vartheta}$, $r \to 0$

$$\left|\int_{\Sigma}\right| \leq \int_{\vartheta=-\pi}^{\pi} \left| \frac{e^{rt\cos\vartheta + irt\sin\vartheta}\, r i e^{i\vartheta}\, d\vartheta}{r^{\alpha+1}\, e^{i\vartheta(\alpha+1)}} \right|$$

$$\leq \int_{\vartheta=-\pi}^{\pi} \frac{e^{rt\cos\vartheta}}{r^{\alpha}}\, d\vartheta \xrightarrow[\text{if } \alpha<0]{} 0$$

** For deformation to work, need $\underline{-1<\alpha<0}$!

(4) At last

$$-\int_{L^+} = \int_{x=r}^{R} \frac{e^{-xt}\, d(-x)}{x^{\alpha+1}\, e^{i\pi(\alpha+1)}} \xrightarrow[r\to 0]{R\to\infty} -e^{-i\pi(\alpha+1)} \underbrace{\int_0^{\infty} e^{-xt} x^{-\alpha-1}\, dx}_{t^{\alpha}\Gamma(-\alpha)}$$

$s = xe^{i\pi} = -x$

$$-\int_{L^-} = -\int_{x=r}^{R} \frac{e^{-xt}\, d(-x)}{x^{\alpha+1} e^{-i\pi(\alpha+1)}} \xrightarrow[r\to 0]{R\to\infty} e^{i\pi(\alpha+1)}\, t^{\alpha}\, \Gamma(-\alpha)$$

$s = xe^{-i\pi} = -x$

The last two integrals converge only if $-1<\alpha<1$.

$$\Rightarrow \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{s^{\alpha+1}}\, ds = \frac{\Gamma(-\alpha)}{\pi} \left(\frac{e^{i\pi(\alpha+1)} - e^{-i\pi(\alpha+1)}}{2i} \right) \Gamma(\alpha+1)$$

$\hookrightarrow$ for $-1<\alpha<0$ $\quad = \underbrace{\frac{\Gamma(\alpha+1)\Gamma(-\alpha)}{\pi} \sin\pi(\alpha+1)}_{=1 \text{ (homework)}}\, t^{\alpha} = t^{\alpha}$

But does this work for all $\alpha > 0$ as well?

(6)

Extension

(a) By Convolution

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \; ; \quad n = 0, 1, \ldots$$

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \; ; \quad -1 < \alpha < 0.$$

$$\mathcal{L}^{-1}\left\{ \frac{(n-1)!}{s^n} \cdot \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \right\} = \int_0^t (t-\tau)^n \tau^\alpha \, d\tau$$

$$\frac{(n-1)!\,\Gamma(\alpha+1)}{s^{\alpha+n+1}} \qquad \sum_{k=0}^{n} \binom{n}{k} t^{n-k} \int_0^t \tau^{\alpha+k} \, d\tau = \ldots, \qquad \int_0^t \tau^{\alpha+k}\,d\tau = \frac{1}{\alpha+k+1} t^{\alpha+k+1}$$

(can finish using combinatorial identities) $\to$ lots of algebra!

(b) Integration by parts (alternative method)

$$\lim_{R\to\infty} \int_{\gamma-iR}^{\gamma+iR} e^{st} s^{-(\alpha+1)} \, ds \equiv I_R(\alpha) \stackrel{\lim_{R\to\infty}}{=} \underbrace{\frac{1}{-\alpha} e^{st}/s^{-\alpha} \Big|_{\gamma-iR}^{\gamma+iR}}_{=0\ (\alpha>0,\,R\to\infty)} + \frac{t}{\alpha} \int_{\gamma-iR}^{\gamma+iR} e^{st} s^{-\alpha} \, ds$$

Suppose $\alpha > 0$.

So: $I(\alpha) = \lim_{R\to\infty} I_R(\alpha) = \frac{t}{\alpha} I(\alpha-1) = (\text{inductively}) = \frac{t^2}{\alpha(\alpha-1)} I(\alpha-2) = \ldots$

$$\ldots = \frac{t^{n+1}}{\alpha(\alpha-1)\cdots(\alpha-n)} I(\alpha-n-1), \text{ provided } \alpha - n > 0$$

If $-1 < \alpha - n - 1 < 0$, then $I(\alpha) = \frac{t^{n+1}}{\alpha(\alpha-1)\cdots(\alpha-n)} \cdot \left| \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st} \, ds}{s^{(\alpha-n-1)+1}} \right|$

$$\Rightarrow I(\alpha) = \frac{t^{n+1}}{\alpha(\alpha-1)\cdots(\alpha-n)} \cdot \frac{t^{\alpha-n-1}}{\Gamma(\alpha-n)}$$

So that $\frac{\Gamma(\alpha+1)}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{s^{\alpha+1}} ds = \underbrace{\left| \frac{\Gamma(\alpha+1)}{\alpha(\alpha-1)\cdots(\alpha-n)\Gamma(\alpha-n)} \right|}_{=1 \text{ by recurrence}} t^\alpha = t^\alpha \; ; \quad \alpha > -1$

So, in general $\frac{\Gamma(\alpha+1)}{2\pi i} \int_\gamma \frac{e^{st}}{s^{\alpha+1}} ds = t^\alpha$, even though the contour can only be deformed for $-1 < \alpha < 0$!