2^{nd} order ODE – variable coefficients

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0$$

($p(z), q(z)$ analytic in simply connected D).

System: $\dfrac{d\underset{\sim}{w}}{dz} = A\underset{\sim}{w}$, $w_1 = w$, $w_2 = \dfrac{dw}{dz}$

$$A = \begin{pmatrix} 0 & 1 \\ -q(z) & -p(z) \end{pmatrix}$$

Picard iteration: $\underset{\sim}{w}^{(1)} = \underset{\sim}{w}_0 = \begin{pmatrix} w(z_0) \\ w'(z_0) \end{pmatrix}$

$$\frac{d\underset{\sim}{w}^{(2)}}{dz} = A\underset{\sim}{w}^{(1)}, \quad \underset{\sim}{w}^{(2)}(z_0) = \underset{\sim}{w}_0$$

$$\cdots$$

$$\frac{d\underset{\sim}{w}^{(n)}}{dz} = A\underset{\sim}{w}^{(n-1)}; \quad \underset{\sim}{w}^{(n)}(z_0) = \underset{\sim}{w}_0$$

or (integral form): $\underset{\sim}{w}^{(n)} = \underset{\sim}{w}_0 + \int_{z_0}^{z} A\underset{\sim}{w}^{(n-1)}(z)\,dz$

Let $|\underset{\sim}{w}| = \sqrt{|w_1|^2 + |w_2|^2}$

(Proof of convergence):

$\forall z \in D: \quad |A| = \max_{\text{all } \underset{\sim}{w}} \dfrac{|A\underset{\sim}{w}|}{|\underset{\sim}{w}|}.$

p, q bounded in D: $\exists M > 0: \quad |A| < M, \ |A w| < M|w|$

then $\quad |\underset{\sim}{w}^{(n)} - \underset{\sim}{w}^{(n-1)}| < M \int_{z_0}^{z} |\underset{\sim}{w}^{(n-1)} - \underset{\sim}{w}^{(n-2)}| \, |dz|$

By analyticity of $\underset{\sim}{w}^{(n)}$, choose straight path.

$n=2$: $\quad \underset{\sim}{w}^{(2)} - \underset{\sim}{w}^{(1)} = \int_{z_0}^{z} A \underset{\sim}{w}_0 \, dz \Rightarrow |\underset{\sim}{w}^{(2)} - \underset{\sim}{w}^{(1)}| < M|z - z_0|$

$n=3$: $\quad |\underset{\sim}{w}^{(3)} - \underset{\sim}{w}^{(2)}| < M \int_{z_0}^{z} |\underset{\sim}{w}^{(2)} - \underset{\sim}{w}^{(1)}| |dz|$

$$< M^2 \int_{z_0}^{z} |z - z_0| dz = M^2 \frac{|z - z_0|^2}{2}$$

Inductively: $\quad |\underset{\sim}{w}^{(n)} - \underset{\sim}{w}^{(n-1)}| < \dfrac{M^{n-1} |z - z_0|^{n-1}}{(n-1)!}.$

Let: $\quad S_n = \underset{\sim}{w}^{(n)} - \underset{\sim}{w}^{(n-1)} \quad \Rightarrow \quad \underset{\sim}{w}^{(n)} = \sum_{k=1}^{n} S_k$; series uniformly convergent since $|S_k| < \dfrac{M^{k-1} |z - z_0|^{k-1}}{(k-1)!}$ (comparison to exp. series).

Hence: $\quad \underset{\sim}{w}(z) = \lim_{n \to \infty} \underset{\sim}{w}^{(n)}(z)$ exists; analytic in $z \in D$.

$\Rightarrow \quad \underset{\sim}{w}(z) = \underset{\sim}{u} + \int_{z_0}^{z} A \underset{\sim}{w}(z) dz$: Equivalent integral equation

$$\therefore \quad \frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \qquad w_0 = a,\ w'_0 = b \qquad (*)$$

has a solution (<u>unique</u> by linearity), analytic in D.

(also: can show analytic in a, b, z_0).

<u>Ordinary point</u>: z_0 : $p(z_0), q(z_0)$ analytic

Assume (with $z_0 = 0$, no loss!)

$p(z) = p_0 + p_1 z + \cdots$, $q(z) = q_0 + q_1 z + \cdots$, $|z| < R$.

$w(z)$ analytic in $|z| < R$: $w(z) = a_0 + a_1 z + \cdots$

Plug into $(*)$: $(2a_2 + 6a_3 z + \cdots) + (p_0 + p_1 z + \cdots)(a_1 + 2a_2 z + \cdots) + (q_0 + q_1 z + \cdots)(a_0 + a_1 z) = 0$

z^0: $2a_2 = -a_1 p_0 - a_0 q_0$

z^1: $6a_3 = -p_1 a_1 - 2a_2 p_0 - a_0 q_1 - a_1 q_0$

$\cdots$

z^n: $n(n-1)a_n = \cdots$

$\Big\}$ a_0, a_1 given $\rightarrow$ find $a_2, a_3, \ldots$

Find $a_n = a_0 u_n(p_0, \cdots, p_{n-2}, q_0, \cdots, q_{n-2}) + a_1 v_n(p_0, \cdots, p_{n-2}, q_0, \cdots, q_{n-2})$

Finally: $w = a_0 \sum_0^\infty u_n z^n + a_1 \sum_0^\infty v_n z^n$

$= a_0 w_1(z) + a_1 w_2(z)$

Note : $a_0 = 1,\ a_1 = 0 \Rightarrow w(0) = 1,\ w'(0) = 0$

Then $w = w_1(z)$: $w_1(0) = 1,\ w_1'(0) = 0$

Similarly $w_2(0) = 0,\ w_2'(0) = 1$

$(w_1(z), w_2(z))$ form a fundamental set.

Equivalent forms for ODE :

* let $P(z) = e^{\int_0^z p(\zeta)d\zeta}$; multiplying through by P

$Pw'' + pPw' + Pqw = 0$; Since $P' = pP$

$\Rightarrow$ $(Pw')' + Qw = 0$ $\quad (Q = Pq)$ self-adjoint

* let $w = u e^{-\frac{1}{2}\int_0^z p(\zeta)d\zeta}$; $w' = (u' - \frac{1}{2}pu)e^{-\frac{1}{2}\int \cdots}$

$w'' = (u'' - pu' + \frac{p^2 u}{4} - \frac{1}{2}p'u)e^{-\frac{1}{2}\int\cdots}$

$\Rightarrow \boxed{u'' + J(z)u = 0}$ $\quad \left(J(z) = q - \frac{p^2}{4} - \frac{p'}{2}\right).$

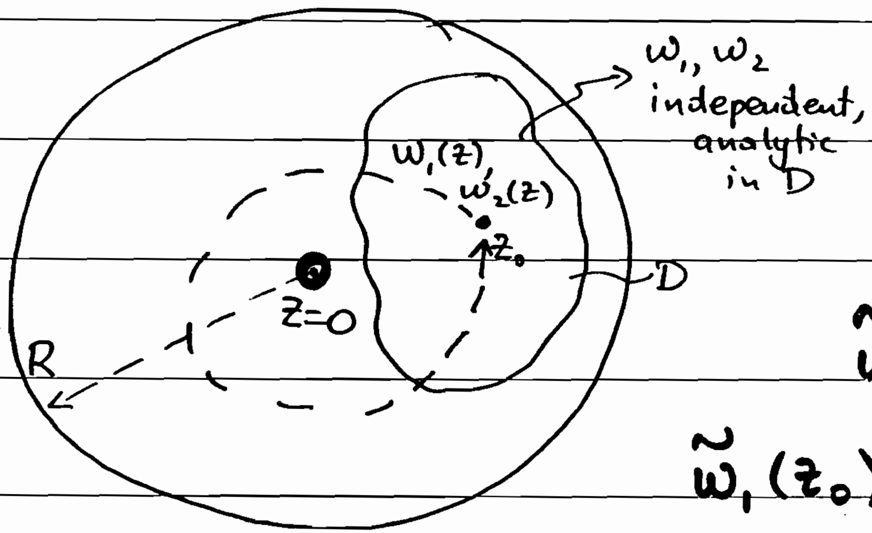
Singular Points : isolated singularities of $p(z)$, $q(z)$.

$z=0$: s.p. ; p, q analytic in $0<|z|<R$.

$\hookrightarrow$ pole of order m (or essential)

no branch points (non-isolated)

Consider neighborhood of z_0 ; (D).



$\omega_1(z)$, $\omega_2(z)$ independent

$\longrightarrow$ Continue analytically to

$$\tilde{\omega}_1 = \omega_1(ze^{2\pi i}), \quad \omega_2(ze^{2\pi i}) = \tilde{\omega}_2$$

$$\tilde{\omega}_1(z_0) = \alpha\, \omega_1(z_0) + \beta\, \omega_2(z_0)$$

$$\tilde{\omega}_2(z_0) = \gamma\, \omega_1(z_0) + \delta\, \omega_2(z_0)$$

$C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$: circuit matrix if C has distinct eigenvalues λ_1, λ_2

can find a pair of solutions u_1, u_2

such that: $\left.\begin{array}{l} u_1 = a w_1 + b w_2 \\ u_2 = c w_1 + d w_2 \end{array}\right\}$:

$$\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

If double eigenvalue: $\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

Example: $w'' - \frac{1}{z} w' + \frac{3}{4z^2} w = 0$

$u_1 = z^{1/2}$, $u_2 = z^{3/2}$

$u_1(ze^{2\pi i}) = -u_1$; $u_2(ze^{2\pi i}) = -u_2$: $C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Circuit matrix diagonal, although eigenvalues are equal

Ex: $w'' + \frac{2}{9z^2} w = 0$

$$u_1 = z^{1/3}, \quad u_2 = z^{2/3}$$

$$u_1(ze^{2\pi i}) = e^{2\pi i/3} u_1$$

$$u_2(ze^{2\pi i}) = e^{4\pi i/3} u_2$$

$$C = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{4\pi i/3} \end{pmatrix}$$

Ex: $w'' - \frac{w'}{z} + \frac{w}{z^2} = 0$: two solutions:

$$u_1 = z, \quad u_2 = \frac{z \log z}{2\pi i}; \quad u_2(ze^{2\pi i}) = u_1 + u_2$$

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

* If $\lambda_1 \neq \lambda_2$, define $f_1(z) = z^{-\sigma_1} u_1$; $f_2(z) = z^{-\sigma_2} u_2$

where $\sigma_1 = \frac{1}{2\pi i} \log \lambda_1$, $\sigma_2 = \frac{1}{2\pi i} \log \lambda_2$

Then: $u_1(ze^{2\pi i}) = \lambda_1 u_1(z)$

$$\Rightarrow f_1(ze^{2\pi i}) = (ze^{2\pi i})^{-\sigma} u_1(ze^{2\pi i}) = \frac{z^{-\sigma_1}}{\cancel{\lambda_1}} \cancel{\lambda_1} u_1 = f_1(z)$$

and similarly $f_2(ze^{2\pi i}) = f_2(z)$

i.e. $f_1(z)$, $f_2(z)$ are single-valued,

and analytic in $0<|z|<R$. Or

$u_1(z) = z^{\sigma_1} f_1(z)$, $u_2(z) = z^{\sigma_2} f_2(z)$

with f_1, f_2 analytic in $0<|z|<R$.

* Double evalue:

$$\left.\begin{aligned} u_1(ze^{2\pi i}) &= \lambda u_1(z) \\ u_2(ze^{2\pi i}) &= u_1(z) + \lambda u_2(z) \end{aligned}\right\}$$

As before, $u_1(z) = z^{\sigma} f(z)$: $\sigma = \frac{1}{2\pi i}\log\lambda$, $f(z)$ anal.

Define: $g(z) = z^{-\sigma}\left(u_2(z) - \frac{\log z}{2\pi i\lambda} u_1(z)\right)$

Then: $g(ze^{2\pi i}) = \frac{z^{-\sigma_1}}{\lambda}\left\{\lambda u_2 + u_1 - \left(\frac{\log z}{2\pi i} + 1\right) u_1\right\}$

$= z^{-\sigma}\left\{u_2(z) - \frac{\log z}{2\pi i\lambda} u_1(z)\right\} = g(z)$

Collect results: (1) $\lambda_1 \neq \lambda_2$: $\sigma_j = \frac{1}{2\pi i} \log \lambda_j$

$$u_j = z^{\sigma_j} f_j(z) \; ; \; f_j \text{ in } 0<|z|<R$$

(i.e. f_j has Laurent series).

(2) $\lambda_1 = \lambda_2 = \lambda$, one e.vector only

$$u_1(z) = z^{\sigma} f(z)$$

$$u_2(z) = z^{\sigma} g(z) + \frac{\log z}{2\pi i \lambda} u_1(z)$$

(f, g analytic in $0<|z|<R$).

*If f_j or f and g have poles of finite order (or zero) at $z=0$, we say that $z=0$ is a <u>regular singularity</u>: the fns. f, g have Laurent expansions $\sum_{-\infty}^{\infty} a_n z^n$ (for regular $\sum_{-m}^{\infty} a_n z^n$).

i.e. if $f_1(z) = \sum_{-m}^{\infty} a_n z^n = z^{-m}(\underset{\neq 0 \text{ at } z=0.}{b_0} + b_1 z + \cdots)$

so $u_1(z) = z^{\sigma_1} z^{-m} F_1(z) = z^{c_1} F_1(z)$

also $u_2(z) = \cdots = z^{c_2} F_2(z)$

(F_1, F_2 analytic, in $|z|<R$, $\neq 0$ at $z=0$).

Similarly: (equal evals):

$$u_1(z) = z^c F(z), \quad u_2(z) = z^{c+n} G(z) + u_1(z) \log z$$
$(n \neq 0)$

$\left(G(z) = a_0 + a_1 z + \cdots \; ; \text{ if } n=0, \quad z^c G(z) - \frac{a_0}{F(0)} z^c F(z) = z^{c+1} H(z) \right.$

so can take $n=1$ to avoid redundancy)

Can show that in order to have a regular singular point, we need $z\,p(z)$, $z^2 q(z)$ finite at $z=0$

Necessary & sufficient condition for regular singular point:

$$p(z) = \frac{1}{z}\left(p_0 + p_1 z + \cdots\right)$$

$$q(z) = \frac{1}{z^2}\left(q_0 + q_1 z + \cdots\right)$$

(p_0, q_0 etc. could be zero)

Frobenius method: let $w = z^c\left(a_0 + a_1 z + \cdots\right)$ $(a_0 \neq 0)$

$$w' = z^{c-1}\left(c\, a_0 + (c+1) a_1 z + \cdots\right)$$

$$w'' = z^{c-2}\left[c(c-1)a_0 + c(c+1)a_1 z + \cdots\right]$$

(eventually, one must show that series for w is convergent)

Rewrite DE: $z^2 w'' + z(zp)w' + (z^2 q)w = 0$

$$\Rightarrow \cancel{a_0}\; z^c\Big\{\left\{c(c-1)a_0 + c(c+1)a_1 z + \cdots\right\} +$$

$$+ \left(p_0 + p_1 z + \cdots\right)\left\{c\, a_0 + (c+1)a_1 z + \cdots\right\} + \left\{q_0 + q_1 z + \cdots\right\}\left\{a_0 + a_1 z\right\} = 0$$

Collecting like powers of z, equating coefficients to zero:

z^c: $a_0\{c(c-1)+p_0c+q_0\}=0$

z^{c+1}: $a_1\{(c+1)c+p_0(c+1)+q_0\}=-(cp_1+q_1)a_0$

$\cdot\ \cdot\ \cdot$

z^{c+n}: $a_n\{(c+n)(c+n-1)+p_0(c+n)+q_0\} = \quad (\dagger)$

$=$ linear combo. of $a_0,\ldots,a_{n-1}$.

Since $a_0 \neq 0$: $\boxed{c(c-1)+p_0c+q_0=0}$ INDICIAL EQUATION

in general two roots, c_1 and c_2

(note: $c_1+c_2=1-p_0$, $c_1c_2=q_0$)

If: $c_1 \neq c_2$, $c_1-c_2 \neq$ integer: two indep. solutions

$$\left.\begin{aligned} w &= z^{c_1}(1+a_1^{(1)}z+\cdots) \\ w &= z^{c_2}(1+a_1^{(2)}z+\cdots) \end{aligned}\right\}:$$ coefficient of a_n in $(\dagger)$ is never zero!

Sum-up: FROBENIUS METHOD

$$w'' + p w' + q w = 0$$

$$p = \frac{1}{z}(p_0 + p_1 z + \cdots)$$

$$q = \frac{1}{z^2}(q_0 + q_1 z + \cdots)$$

$$w = z^c(a_0 + a_1 z + \cdots)$$

Substitution gives

z^c : $\quad a_0 \{ c(c-1) + p_0 c + q_0 \} = 0$ INDICIAL EQUATION

z^{c+1} : $\quad a_1 \{ (c+1)c + p_0(c+1) + q_0 \} = -(c p_1 + q_1) a_0$

(#) z^{c+n} : $\quad a_n \{ (c+n)(c+n-1) + p_0(c+n) + q_0 \} =$ linear combo. of $a_0, \ldots, a_{n-1}$

INDICIAL EQUATION: $\quad c(c-1) + p_0 c + q_0 = 0 \qquad \begin{cases} c_1 + c_2 = 1 - p_0 \\ c_1 c_2 = q_0 \end{cases}$

If roots are c_1, $c_1 = c_2 + n$, then expansion for c_2 begins with a_n (arbitrary) while $a_0 = \cdots = a_{n-1} = 0$ (i.e. no new info.)

If the R.H.S. of (P) for $n=N$ is also zero, then a_n remains arbitrary. Choosing a value for a_n, recursion can be continued. There results

$$u_2(z) = z^{c_2}\left(a_0 + a_1 z + \dots\} + a_{N-1} z^{N-1} + a_{N+1} z^{N+1} + \dots\right)$$
$$+ z^{c_2}\left(b_N z^N + \dots\right)$$

The second term is a constant multiple of u_1. The u_1 and u_2 obtained this way are independent (the circuit matrix has equal eigenvalues but it is diagonal).

BUT: if the RHS of (P) for $n=N$ is not zero, method fails to give a second solution (see case (3)).

Cases (1) $c_1 \neq c_2$, $c_1 - c_2 \neq$ integer

$\therefore$ method gives two independent sols.

$$w_j = z^{c_j}\left(1 + a_1^{(j)} z + \cdots\right), \quad j=1,2.$$

(2) $c_1 = c_2 + N$, N integer

Now $u_1(z) = z^{c_1}\left(a_0^{(1)} + a_1^{(1)} z + \cdots\right)$ is always a solution. The $a_n^{(1)}$ are well determined because the coefficient of a_n in (†) never vanishes. Let us guess (naturally) the second soln.:

$$u_2(z) = z^{c_2}\left(a_0^{(2)} + a_1^{(2)} z + \cdots\right)$$

Since $c_2 + N = c_1$ is a root of the indicial equation:

$$(c_2+N)(c_2+N-1) + p_0(c_2+N) + q_0 = 0$$

i.e. (†) with $N=n$ gives: $a_N \cdot 0 = \text{RHS}\left(\begin{array}{c}\text{linear combination of} \\ a_0, a_1, \ldots, a_{N-1}\end{array}\right)$

(3) Double root, $c_1 = c_2$ (or $c_1 = c_2 + N$ with RHS. of f, $n = N$, not zero)

One solution $u = z^c (1 + a_1 z + \cdots)$

(or, use $c = c_1$ in case $c_1 = c_2 + N$, $N \geq 0$).

Look for second solution in the form

$w = vu$, v unknown ("reduction of order")

Then $w' = v'u + u'v$; $w'' = v''u + 2v'u' + vu''$

Substituting into equ:

$$z^2 u v'' + v'(2z^2 u' + z(p_0 + p_1 z + \cdots)u) = 0$$

or

$$\frac{v''}{v'} = -\frac{2u'}{u} - \frac{p_0}{z} - (p_1 + p_2 z + \cdots) = -\frac{2u'}{u} - \frac{p_0}{z} - \phi(z).$$

Integrate once:

$$v' = \frac{z^{-p_0}}{u^2} e^{-\int \phi(z)\,dz}$$

When c is a double root, the formula for the sum of the roots gives

$$p_0 = 1 - 2c.$$

Recall that $u = z^c \psi(z)$, $\psi(z)$ analytic.

Hence:

$$v' = \frac{z^{2c-1}}{z^{2c}\psi^2} e^{-\int \phi(z)\,dz} = \frac{1}{z}\Phi(z)$$

where $\Phi(z)$ is analytic at $z=0$ and $\Phi(0)=1$.

Then $\frac{1}{z}\Phi(z)$ can be written as $\frac{1}{z} + \alpha_1 + \alpha_2 z + \cdots$

Hence $\quad v = \log z + \alpha_1 z + \frac{\alpha_2}{2} z^2 + \cdots$

$$= \log z + z\,\Psi(z)$$

and $\quad w = u \log z + z^{c+1}\chi(z)$

—— . ——

When $c_1 = c_2 + N$, (4) rhs $\neq 0$ for $n = N$:

Again, let $w = v u$

Now $$\frac{v''}{v'} = -\frac{2u'}{u} - \frac{p_0}{z} - p_1 - p_2 z - \dots$$

(as before)

$$\Rightarrow v' = \frac{z^{-p_0}}{u^2}\Phi(z) = \frac{z^{-p_0}}{z^{2c_1}}\psi(z)$$

($\Psi(z)$ analytic at $z=0$, $\Psi(0)=1$).

Now, sum formula gives $p_0 = 1 + N - 2c_1$

so that

$$v' = \frac{1}{z^{N+1}}\Psi(z)$$

Integrating:

$$v = \frac{1}{z^N} + \frac{a_1}{z^{N-1}} + \dots + \frac{a_{N-2}}{z^2} + \log z + \chi(z)$$

(a multiplicative const α was omitted).

Behavior at $z=\infty$: if p, q analytic for $|z|>R$, is $z=\infty$ a regular or a singular point?

Answer: NO!

Ex: $w''+w=0 \Rightarrow w=A\cos z+B\sin z$, $z=\infty$ is essential singularity.

Change variables: $z=1/\zeta$: study behavior near $\zeta=0$.

$$\frac{dw}{dz}=\frac{dw}{d\zeta}\frac{d\zeta}{dz}=-\zeta^2\frac{dw}{d\zeta} \quad \text{and}$$

$$\frac{d^2w}{dz^2}=-\zeta^2\frac{d}{d\zeta}\left(-\zeta^2\frac{dw}{d\zeta}\right)=\zeta^4\frac{d^2w}{d\zeta^2}+2\zeta^3\frac{dw}{d\zeta}$$

$$\Rightarrow \zeta^4\frac{d^2w}{d\zeta^2}+2\zeta^3\frac{dw}{d\zeta}-p\left(\frac{1}{\zeta}\right)\zeta^2\frac{dw}{d\zeta}+q\left(\frac{1}{\zeta}\right)w=0$$

$$\Rightarrow \quad \frac{d^2 w}{d\zeta^2} - \left(\frac{P}{\zeta^2} - \frac{2}{\zeta}\right)\frac{dw}{d\zeta} + \frac{Q}{\zeta^4}\, w = 0$$

$$P(\zeta) = p\left(\tfrac{1}{\zeta}\right), \quad Q(\zeta) = q\left(1/\zeta\right).$$

$\Rightarrow$ $z=\infty$ ordinary if $\left.\begin{matrix} \dfrac{P}{\zeta^2} - \dfrac{2}{\zeta} \\ Q/\zeta^4 \end{matrix}\right\}$ analytic at $\zeta = 0$.

(or, equivalently, if $z^2 p(z) - 2z$, $z^4 q(z)$ bounded at ∞)

$z=\infty$ regular singularity if $\dfrac{P}{\zeta^2} - 2$, $\dfrac{Q}{\zeta^2}$ analytic at $\zeta=0$

(or $|z^2 q| < \infty$, $|zp| < \infty$ as $|z| \to \infty$.).

Solutions are found in ζ variable and translated to z.

Inhomogeneous Equation:

$$w'' + pw' + qw = r$$

Let $\quad r = z^{\gamma} \sum_{0}^{\infty} r_n z^n$

If $z=0$ is ordinary, then

$$w = z^{\gamma+2}(a_0 + a_1 z + \cdots)$$

If $z=0$ regular s.p. with

$$p = \frac{1}{z}(p_0 + p_1 z + \cdots), \quad q = \frac{1}{z^2}(q_0 + q_1 z + \cdots)$$

then
$$z^2 w'' + z(p_0 + p_1 z + \cdots)w' + (q_0 + q_1 z + \cdots)w = z^{\gamma+2} \sum_{0}^{\infty} r_n z^n$$

Still, $w = z^{\gamma+2} \sum_{0}^{\infty} a_n z^n$ works, provided $\gamma+2, \ldots \gamma+n$, are not roots of indicial equ.

Ex: $$w'' - \frac{2w'}{z} + \frac{2w}{z^2} = z$$

The roots of indicial eqn. are 1, 2.

But $\gamma = 1$. Hence

$$w = z^3(a_0 + a_1 z + \cdots) \quad \text{works.}$$

Ex: $$w'' - \frac{2w'}{z} + \frac{2w}{z^2} = 1$$

$c_1 = 1$, $c_2 = 2$, $\gamma = 0$. Series starting with $w = z^2$ fails. But if we try

$$w = z^2(a_0 + a_1 z + \cdots) \log z$$

we find

$$w = z^2 \log z$$

is particular solution.