LEGENDRE FUNCTIONS & POLYNOMIALS

Consider Laplace eqn. - 3d spherical coords

$$\partial_r\left(r^2 \sin\vartheta \frac{\partial u}{\partial r}\right) + \partial_\vartheta\left(\sin\vartheta \frac{\partial u}{\partial \vartheta}\right) + \frac{1}{\sin\vartheta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

Then $u = r^\nu y(x)$, $x = \cos\vartheta$ solution if

$$\boxed{(1-x^2)\,y'' - 2xy' + \nu(\nu+1)\,y = 0} \qquad (1)$$

Regular singular points at $x = \pm 1, \infty$.

$x = \pm 1$: indicial equation $c^2 = 0$ (one solution regular, other has log singularity

(in fact solution that is regular at $+1$ is singular at -1 etc.)

Solution regular at both $x = +1, -1$: $\nu = n$ or $(n+1)$. Then solutions are POLYNOMIALS in x, $P_n(x)$, of degree n.

(normalize: $P_n(1) = 1$: Legendre polynomials).

Most physical problems: soln. regular at $\vartheta = 0, \pi$ $(x = \pm 1)$.

H. Hochstadt, "The function of Math. Physics";

$P_1 = \frac{w'}{w}(1-x^2) - 2x = Ax+B$

$P_n(x) = \frac{1}{w}\left(\frac{d}{dx}\right)^n w(x)(1-x^2)^n$

$[-1,1]$ $\Rightarrow \int_{-1}^{1} w(x)x^k p_n(x)\,dx \Leftrightarrow w = (1-x)^\alpha(1-x)^\beta$

$(0,\infty)$: $p_n = \frac{1}{w}\left(\frac{d}{dx}\right)^n wx^n$: $w = x^\alpha e^{-x}$

$\hookrightarrow \frac{w'}{w}x + 1 = B - x$ $\swarrow A=-1$

$(-\infty,\infty)$ $p_n = \frac{1}{w}\left(\frac{d}{dx}\right)^n w$: $\frac{w'}{w} = -x$; $w = e^{-x^2/2}$

$\int_{-\infty}^{\infty} w\, x^k p_n = 0,\ 0 \le k < n$

Rodriguez formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

(O. Rodrigues, "Memoire sur l'attraction des spheroides." Corr. Ecole (Roy.) Polytechnique III (1816))

▼ since $y(x) = (x^2-1)^n$ satisfies

G. Sansone, "Orthogonal functions"

$$(1-x^2)y'' + 2(n-1)xy' + 2ny = 0,$$

differentiate each term n times, using the Leibnitz rule

$$\frac{d^n}{dx^n}(uv) = uv^{(n)} + nu'v^{(n-1)} + \frac{n(n-1)}{2}u''v^{(n-2)} + \dots + u^{(n)}v,$$

Get: $\left((1-x^2)y^{(n)''} - 2nxy^{(n)'} - n(n-1)y^{(n)}\right) +$

$\left(2(n-1)xy^{(n)'} + 2n(n-1)y^{(n)}\right) + 2ny^{(n)} = 0$. This simplifies

to: $(1-x^2)y^{(n)''} - 2xy^{(n)'} + n(n+1)y^{(n)} = 0$. So $y^{(n)} = \alpha P_n(x)$.

Normalization: $y^{(n)}(x) = \frac{d^n}{dx^n}(x+1)^n(x-1)^n = (x+1)^n \cdot n! + n(x+1)^{n-1} n!(x-1) + \dots$

so $y^{(n)}(1) = 2^n n!$. ▲

17.3

Similarly show $y^{(n)}(-1) = (-2)^n n!$

i.e. $P_n(-1) = (-1)^n$.

(could also conclude this from evenness of $P_{2n}(x)$, oddness of $P_{2n+1}(x)$).

<u>Orthogonality</u>: let $n > m$

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{1}{2^{n+m} n! m!} \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m dx$$

$$= \frac{(-1)^n}{2^{n+m} n! m!} \int_{-1}^{1} (x^2-1)^n \underbrace{\frac{d^{m+n}}{dx^{m+n}}}_{m+n \text{ differentiations} > 2m} \underbrace{(x^2-1)^m}_{\text{degree } 2m} dx = 0 \quad : \quad \int_{-1}^{1} P_n P_m\, dx = 0 \quad n \neq m$$

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{(-1)^n}{2^n (n!)^2} \int_{-1}^{1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}} (x^2-1)^n dx = \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (1-x^2)^n dx$$

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let $I_n = \int_{-1}^{1} (1-x^2)^n dx =$ (by parts)

$$= \left(x(1-x^2)\right|_{-1}^{1} + \int_{-1}^{1} 2n\, x^2 (1-x^2)^{n-1} dx$$

$$= -2n\int_{-1}^{1} (1-x^2)^n dx + 2n\int_{-1}^{1} (1-x^2)^{n-1} dx$$

i.e. $I_n = -2n I_n + 2n I_{n-1}$

$$\Rightarrow I_n = \frac{2n}{2n+1} I_{n-1} = \cdots = \frac{2^{2n}(n!)^2}{(2n)!} \cdot \frac{2 = \int_{-1}^{1} dx}{2n+1}$$

Note: $\int_{-1}^{1} dx = 2 = I_0$

Finally $\int_{-1}^{1} P_n^2(x)\,dx = \dfrac{2}{2n+1}$

Expansion theorem : $f(x)$ continuous, $x \in (-1,1)$

$$f(x) = \sum_0^\infty a_n P_n(x) \qquad (2)$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx$$

(uniform convergence where $f(x)$ smooth)

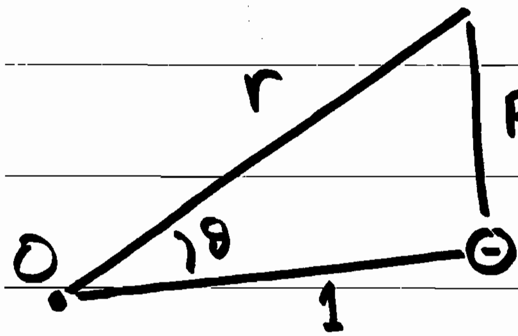
[Weierstrass approx. theorem: $f(x)$ continuous on $[a,b]$

$\Rightarrow \forall \varepsilon > 0 \ \exists \ N = N(\varepsilon)$: there is a poly $p(x)$

of degree N such that $|f(x) - p(x)| < \varepsilon$ on $[a,b]$]

Generating function:

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_0^{\infty}{}' P_n(x) t^n \qquad (3)$$



$R = \sqrt{r^2 + 1 - 2r\cos\theta}$

Potential of point charge is $\frac{1}{R}$, R = distance from charge

i.e. $\nabla^2\left(\frac{1}{R}\right) = 0$, $R \neq 0$.

Adopt polar coords. (r, θ, ϕ) with origin at distance 1 from charge. As we have seen (from (1)), any solution of Laplace's equation that is smooth in (r, θ) (indep. of ϕ) in $0 \leq r < 1$ can be expressed by series

$$\sum_0^\infty a_n r^n P_n(\cos\theta)$$

(since solns. of (1): $u = r^\nu$ ..(x) ; $\nu = n$, $x = \cos\theta$, $y = P_n(x)$)

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$$\Rightarrow \frac{1}{R} = \frac{1}{\sqrt{r^2+1-2r\cos\theta}} = \sum_0^\infty a_n r^n P_n(\cos\theta)$$

$r<1.$

Set $\theta=0$: $\frac{1}{1-r} = \sum_0^\infty a_n r^n P_n(1) = \sum_0^\infty a_n r^n$

$\Rightarrow a_n = 1$

so: $$\frac{1}{\sqrt{r^2+1-2rx}} = \sum_0^\infty r^n P_n(x) = G(x,t)$$

Recurrence relns. : $(1-2xt+t^2)\, G^2(x,t) = 1$

$\frac{d}{dt}$: $(1-2xt+t^2)\frac{\partial G}{\partial t} + (t-x)\, G = 0$

Since $G = \sum t^n P_n(x)$:

$$(2n+1)\, x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) \quad (4)$$

$\frac{d}{dx}$: find $$P_n(x) = P'_{n+1}(x) - 2x P'_n(x) + P'_{n-1}(x) \quad (5)$$

17.8

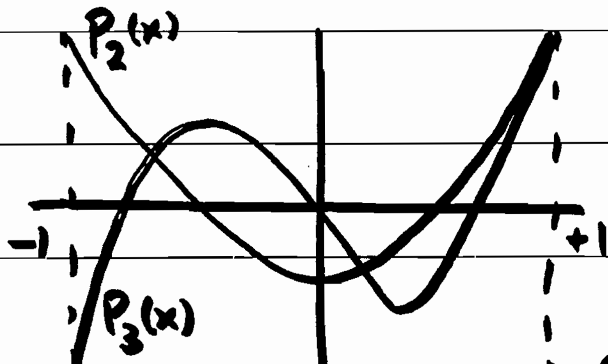
Claim: (i) P_{n+1} has one zero between consecutive zeros of P_n

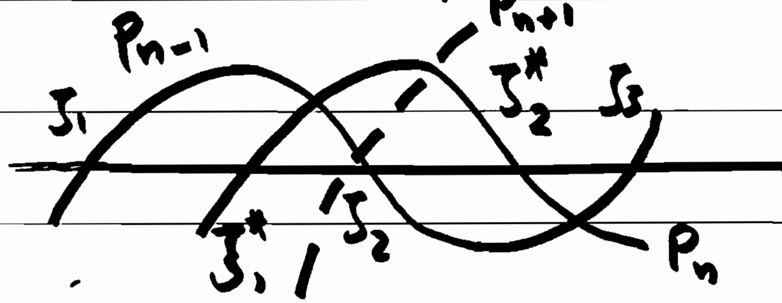
(ii) $P_n(x)$ has n distinct zeros in $(-1,1)$

Induction: suppose $P_u(x)$ has u distinct roots in $(-1,1)$, $u \le n$;

also P_n has one root bet. consec. roots of P_{n-1}. Let ξ_1, ξ_2, ξ_3 be three consecutive roots of P_{n-1} (distinct by assumption)

P_n has ξ_1^* in (ξ_1, ξ_2) and ξ_2^* in (ξ_2, ξ_3).





Use recurrence (4):

$$\left(P_{n-1}(\xi_1) = P_{n-1}(\xi_2) = P_{n-1}(\xi_3) = 0 \right)$$

$$P_n(\xi_1^*) = P_n(\xi_2^*) = 0$$

$$(n+1)\,p_{n+1}(\zeta_1^*) + n\,\overbrace{p_{n-1}(\zeta_1^*)}^{(+)} = 0$$

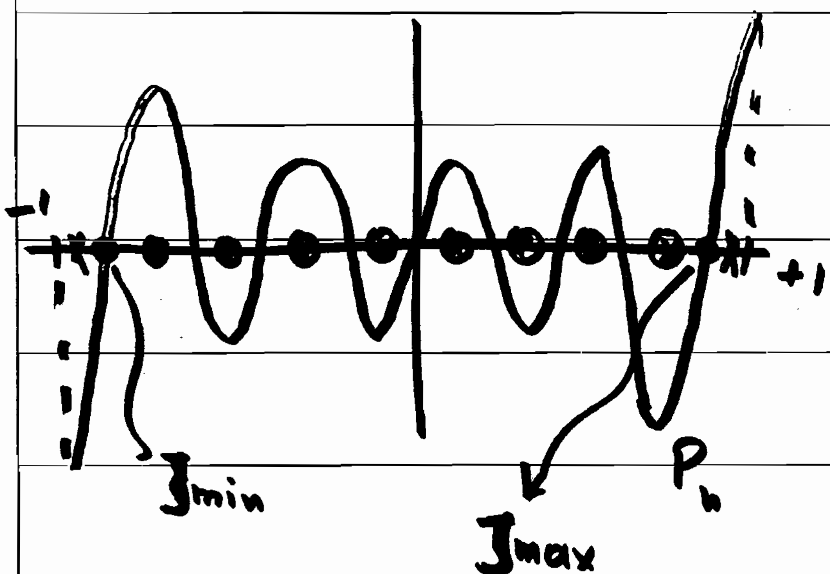
$$(n+1)\,p_{n+1}(\zeta_2^*) + n\,\underbrace{p_{n-1}(\zeta_2^*)}_{(-)} = 0$$

Hence p_{n+1}, p_{n-1} have opposite signs at ζ_1^*, ζ_2^*. Since p_{n-1} changes sign between ζ_1^*, ζ_2^*, so does p_{n+1}. $\Rightarrow$ p_{n+1} has at least one root between ζ_1^*, ζ_2^*. Since p_n has n roots, this gives us $(n-1)$ roots of p_{n+1}. Need two more!

17.10

Can show that $P_{n+1}(x)$ has one root in $\xi_{max} < x < 1$ and another in $-1 < x < \xi_{min}$.

Since $2 + (n-1) = n+1$ this gives all roots of $P_{n+1}(x)$; so there is exactly one root of $P_{n+1}(x)$ between two consecutive roots of $P_n(x)$.



⊗ : roots accounted for $(\bullet)\ n-1$

x : remaining roots