Fourier Series (assume convergence!)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (*)$$

$$f(x) = f(x+2\pi) \; : \; \text{periodicity}$$

Orthogonality

$m \neq n$: $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx$

all m, n $\displaystyle = \int_0^{2\pi} \cos mx \sin nx \, dx = 0$

$m \neq n$: $\displaystyle\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi$ $(n \neq 0)$

$$\Rightarrow \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

procedure o.k. if convergence uniform (interchange $\Sigma, \int$).

<u>Question</u>: given $f(x)$, under what condition is the Fourier series $(*)$ equal to $f(x)$ at a given x?

(Fourier series) <u>Theorem</u>: if $f(x)$ is piecewise differentiable and absolutely integrable on $-\pi \leq x \leq \pi$, then the Fourier series converges uniformly on intervals where $f(x)$ is differentiable. In this case, if we define

$$F(x) := \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \quad {}^{*}$$

we have $F(x) = f(x)$. If $f(x)$ has a simple jump at $x=a$ then the Fourier series converges to $F(a) = \frac{1}{2}(f(a^+) + f(a^-))$, but the convergence is not uniform in any neighborhood of $x=a$.

<u>Note</u>: we define the partial sum

$$F_N(x) := \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx$$

* $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx$, $k = 0, 1, 2, \ldots$

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<u>Fourier cosine series</u> : $f(x) = f(-x)$: even

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f\cos\frac{n\pi x}{L}\,dx$$

$$b_n = 0$$

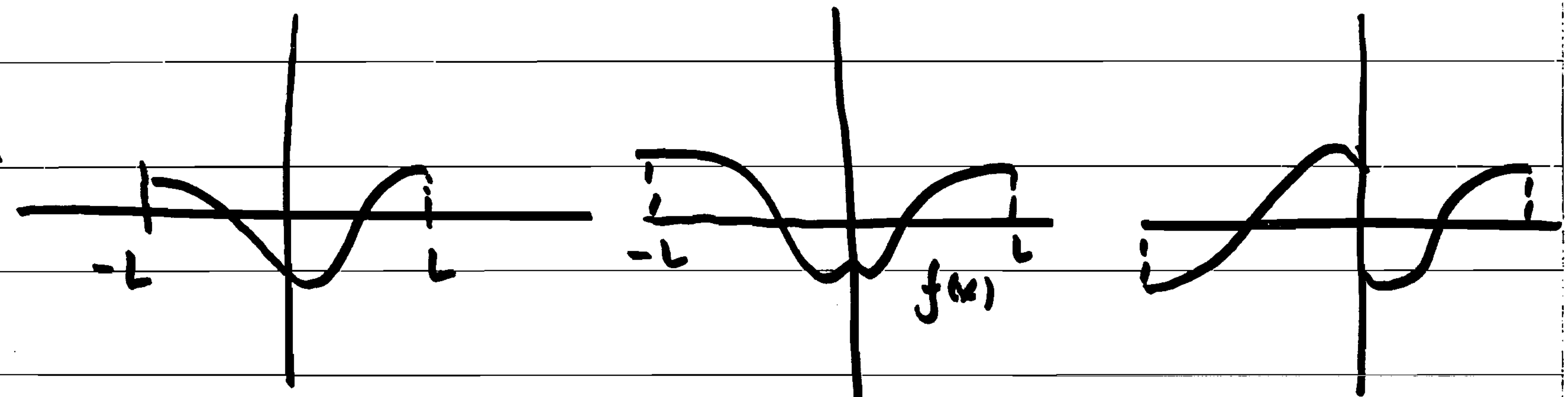
(1) $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

<u>Fourier sine series</u>: $f(x) = -f(-x)$ odd

$$a_n = 0, \quad b_n = \frac{1}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

(2) $$f(x) = \sum_{k=1}^{\infty} f_n \sin\frac{n\pi x}{L}$$

(1):



f(x) even extension odd extension

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<u>Complex Fourier series</u>

let $f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx + b_n \sin nx$

(period 2π)

since $\cos nx = \frac{1}{2}(e^{ix} + e^{-ix})$, $\sin nx = \frac{1}{2i}(e^{ix} - e^{-ix})$

$$\Rightarrow f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \begin{cases} \frac{1}{2}(a_n + ib_n), & n > 0 \\ \frac{1}{2}(a_{-n} - ib_{-n}), & n < 0 \end{cases}$$

$$c_0 = \frac{a_0}{2}$$

$$\left(\int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx = 2\pi \delta_{nm} \right)$$

↳ notice form: $\langle \phi, \psi \rangle = \int \phi \psi^* \, dx$

$$C_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} \, dx$$

Parseval's theorem: If $\int_{-\pi}^{\pi} f^2(x)\,dx < \infty$:

$$\int_{-\pi}^{\pi} f^2(x)\,dx = \int_{-\pi}^{\pi}\left(\sum c_k e^{ikx}\right)\left(\sum \bar{c}_k e^{-ikx}\right)dx$$

in general $|f|^2 = ff^*$

$$= 2\pi \sum_{-\infty}^{\infty} c_k \bar{c}_k = \pi\left[\frac{a_0^2}{2} + \sum_1^{\infty}(a_n^2 + b_n^2)\right]$$

Partial Sums:

How can we best approximate $f(x)$ by $F_N(x)$?

Can show $$\int_0^{2\pi}\left[f(x) - \frac{\alpha_0}{2} - \sum_1^N(\alpha_k \cos kx + \beta_k \sin kx)\right]^2 dx$$

becomes minimal over all choices of α_k, β_k if we set

$\alpha_k = a_k$, $\beta_k = b_k$, $k = 0, 1, \ldots$ (F_N gives the best

mean square approximation):

Indeed, if we substitute for $f(x)$ its Fourier series, we have

$$0 \le \int_{-\pi}^{\pi} \left[f(x) - \frac{\alpha_0}{2} - \sum_1^N (\alpha_k \cos kx + \beta_k \sin kx) \right]^2 dx \underset{\text{(by Parseval)}}{=}$$

$$= \frac{(a_0 - \alpha_0)^2}{2} + \sum_1^N \left[(a_k - \alpha_k)^2 + (b_k - \beta_k)^2 \right] \qquad (*)$$
$$+ \sum_{N+1}^{\infty} (a_k^2 + b_k^2)$$

since all factors are positive, minimum (which is generally positive), is achieved for $a_k = \alpha_k$, $b_k = \beta_k$.

Expanding expression in $(*)$: (since $\int f^2 dx = \pi a_0^2$)

$$0 \le \frac{\pi}{2} a_0^2 + \frac{\pi}{2} \alpha_0^2 - \pi a_0 \alpha_0 + \pi \sum \alpha_k^2 + \beta_k^2 - 2\pi \sum a_k \alpha_k + b_k \beta_k$$

$\Rightarrow$ set $a_k = \alpha_k, b_k = \beta_k$: $\quad \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 dx \ge \sum_1^N (a_k^2 + b_k^2) + \frac{a_0^2}{2}$

(easy from Parseval).

Convergence in the mean:

if $\int f^2 dx < \infty \Rightarrow \sum_1^{\infty}(a_k^2 + b_k^2) + \frac{a_0^2}{2} < \infty$

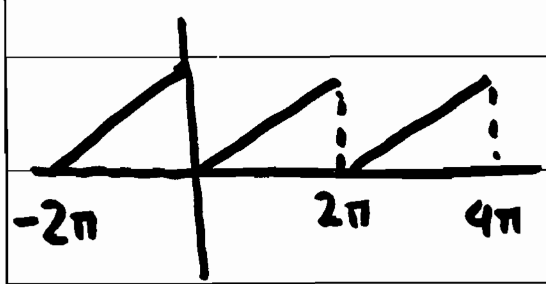
↳ series convergent.

$$\Rightarrow \int_{-\pi}^{\pi} [f - F_N]^2 dx = \sum_{N+1}^{\infty}(a_k^2 + b_k^2) \xrightarrow{N\to\infty} 0$$

<u>convergence in the mean</u>

$\Rightarrow (\cos nx, \sin nx)$ <u>complete</u>.

<u>Examples</u> (1) $f(x) = x$, $0 < x < 2\pi$ (periodic extension)



(upshifted odd function, i.e. $f(x) = \pi + \text{odd}$)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\, dx = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \cos nx = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x\, dx =$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\right|_0^{2\pi} + \frac{1}{\pi}\int_0^{2\pi}\frac{\cos nx}{n}dx = -\frac{2}{n}$$

$$f(x) = \pi - 2\sum_1^{\infty} \frac{\sin nx}{n}, \quad 0<x<2\pi$$

$$\left(F(0) = F(2\pi) = \pi\right)$$

Parseval: $\displaystyle\int_0^{2\pi} x^2\,dx = \frac{8\pi^3}{3} = \pi\left(2\pi^2 + \sum_1^{\infty}\frac{1}{n^2}\right)$

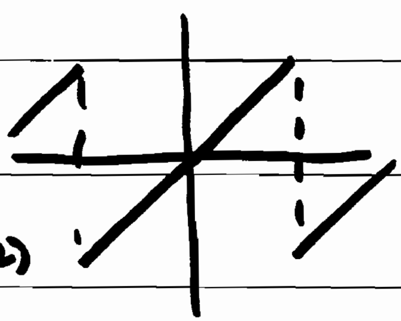
or $\displaystyle\sum_1^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

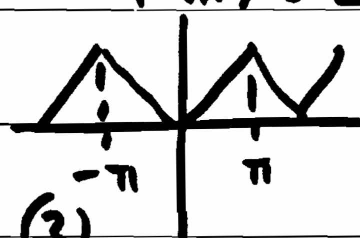
<u>Note</u>: $f'(x) = 1$ but

$\displaystyle 1 = -2\sum_1^{\infty}\cos nx$ meaningless!

<u>Ex. 2</u>: $f(x) = x$ in $-\pi<x<\pi$

$$F(x) = 2\sum_1^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

(2) 

(3) 

<u>Ex. 3</u>: $f(x) = |x|$, $-\pi<x<\pi$: $\displaystyle F(x) = \frac{\pi}{4} + 2\sum_{\substack{1\\ n \text{ odd}}}^{\infty}\frac{(-1)^{n}}{\pi n^2}\cos nx$

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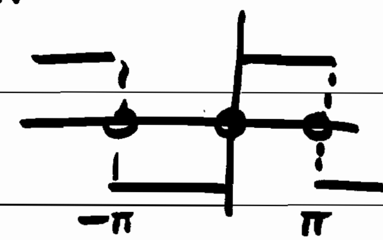
Now $F'(x) = 2\sum_{\substack{1 \\ n \text{ odd}}}^{\infty} \frac{\cancel{\text{[illegible]}}}{\pi n} \sin nx$

F'



Q: When can the Fourier series be differentiated term-by-term?

A: when $f(x)$ is continuous in $0 \le x \le L$, $f(0) = f(L)$, and $f'(x)$ has a convergent Fourier series.

Integration is possible if $f(x)$ is piecewise continuous and integrable (even if Fourier series does not converge).

TRUNCATION ERROR (approximation by partial sums)

Let: $f(x)$ continuous, 2π-periodic

$f'(x)$ has Fourier series:

$$f'(x) = \sum_1^{\infty} n b_n \cos nx - n a_n \sin nx$$

$$\left(f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx + b_n \sin nx \right)$$

Parseval: $\frac{1}{\pi}\int_0^{2\pi} f'^2(x)\,dx = \sum_1^{N} n^2 (a_n^2 + b_n^2)$

Let $S_N(x) = \frac{a_0}{2} + \sum_1^{N} (a_n \cos nx + b_n \sin nx)$

$$|S_M(x) - S_N(x)| = \sum_{N+1}^{M} |a_n \cos nx + b_n \sin nx|$$

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Cauchy-Schwartz inequality : α_n, β_n given

$$\left|\sum_1^N \alpha_n \beta_n\right| \le \left(\sum_1^N \alpha_n^2\right)^{1/2}\left(\sum_1^N \beta_n^2\right)^{1/2}$$

So:

$$|a_n \cos nx + b_n \sin nx| \le (a_n^2+b_n^2)^{1/2}$$

$$(a_n^2+b_n^2)^{1/2}(\underbrace{\cos^2 nx + \sin^2 nx}_{=1})^{1/2}$$

So: $$|S_M - S_N| \le \sum_{N+1}^{M} (a_n^2+b_n^2)^{1/2}$$

$$= \sum_{N+1}^{M}\left(\frac{1}{n}\right)\left\{n^2(a_n^2+b_n^2)\right\}^{1/2} \le \left(\sum_{N+1}^{M}\frac{1}{n^2}\right)^{1/2}\left(\sum_{N+1}^{M} n^2(a_n^2+b_n^2)\right)^{1/2}$$

But, since $\frac{1}{\pi}\int_0^{2\pi} f'^2\,dx \ge \sum_{N+1}^{M} n^2(a_n^2+b_n^2)$ we find

$$|S_M - S_N| \le \frac{1}{\sqrt{\pi}}\left(\sum_{N+1}^{M}\frac{1}{n^2}\right)^{1/2}\left(\int_0^{2\pi} f'^2\,dx\right)^{1/2}$$

and, as $M \to \infty$: $$|f(x) - S_N(x)| \le \frac{1}{\sqrt{\pi}}\left(\sum_{N+1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\int_0^{2\pi} f'^2\,dx\right)^{1/2}$$

If $f(x)$ has jump at $x=a$, then Fourier series of $f(x)$ does not converge uniformly to $f(x)$ near a:

<u>Gibb's phenomenon</u> $\quad f(x) = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$

Fourier series: $f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \cdots\right)$

Using the identity (Hwk. 1): $\cos\tau + \cos 3\tau + \cdots + \cos(2m-1)\tau = \frac{1}{2}\frac{\sin 2m\tau}{\sin\tau}$,

integrating both sides:

$$S_{2m-1}(x) \equiv \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2m-1)x}{2m-1}\right)$$

$$= \frac{2}{\pi}\int_0^x \frac{\sin 2m\tau}{\sin\tau}\, d\tau$$

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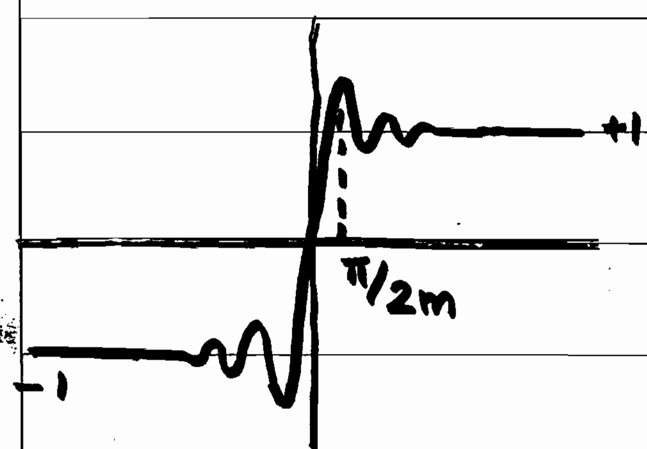
Now: $S'_{2m-1}(x) = \frac{2}{\pi}\,\frac{\sin 2mx}{\sin x}$, so that $S_{2m-1}(x)$ has extrema when $\sin 2mx = 0$ or $x = \frac{k\pi}{2m}$; $k=1$ gives the tallest max. We find

$$S_{2m-1}\left(\frac{\pi}{2m}\right) = \frac{2}{\pi}\int_0^{\frac{\pi}{2m}} \frac{\sin 2m\tau}{\sin\tau}\,d\tau =$$

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{2m\sin t/2m}\,dt \xrightarrow{m\to\infty} \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt = 1.179$$

(If the convergence were uniform, this limit would be 1.).



(Since $\int_0^{\pi} S_{2m-1}^2(x)\,dx \to \int_0^{\pi} f(x)\,dx = \pi$, overshoot must be balanced by undershoot etc.)

Ex: Solve by separation of variables (SV)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \le r < 1,$$

$$u(1,\theta) = f(\theta)$$

let $u = R(r)\Theta(\theta)$; find

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0 \Rightarrow$$

separation constant; set negative so Θ oscillatory (periodic)

$$\Theta''/\Theta = -k^2 = -r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right)$$ Let k = integer: 2π-periodic

$\Theta_k(\theta) = A_k\cos k\theta + B_k\sin k\theta$; $k = 1, 2, \dots$ $\quad \Theta_0(\theta) = \frac{A_0}{2}$ ($\Theta_0'' = 0$; second solution not periodic)

Then $R'' + \frac{1}{r}R' - \frac{k^2}{r^2}R = 0$; $R(0)$ bounded, R smooth at 0.

Let $R = r^\lambda$: $\lambda^2 \cancel{-\lambda+\lambda} - k^2 = 0 \Rightarrow \lambda = \pm k$; choose $\lambda = +k$

$R_k(r) = r^k$. So $u(r,\theta) = \frac{A_0}{2} + \sum_{k=1}^{\infty} r^k\left(A_k\cos k\theta + B_k\sin k\theta\right)$.

At $r=1$, $u(1,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{1}^{\infty} a_k\cos k\theta + b_k\sin k\theta$

Ex: same as before, but now

$$u_r(1,\theta) = f(\theta).$$

Now $\displaystyle u_r(r,\theta) = \sum_{k=1}^{\infty} k r^{k-1}(A_k \cos k\theta + B_k \sin k\theta)$

$\displaystyle = \frac{a_0}{2} + \sum_{1}^{\infty} a_k \cos k\theta + b_k \sin k\theta$

No solution if $\displaystyle a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta \neq 0$

Can also see this as follows:

$$\iint_{r\leq 1} \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) r\,dr\,d\theta = 0 \quad (\text{D.E.})$$

$$\int_{\theta=0}^{2\pi} d\theta \left(\int_{r=0}^{1} (r u_{rr} + u_r)\,dr\right) + \int_{r=0}^{1} \frac{1}{r}dr \left(\underbrace{\int_0^{2\pi} u_{\theta\theta}\,d\theta}_{=0}\right) = 0$$

$=0$ since $u(r,\theta)$ 2π-periodic in θ

$$\int_{r=0}^{1} (r u_r)'\,dr = r u_r\Big|_0^1 = u_r(1,\theta)$$

$$\int_{\theta=0}^{2\pi} u_r(1,\theta)\,d\theta = \int_{\theta=0}^{2\pi} f(\theta)\,d\theta = 0$$

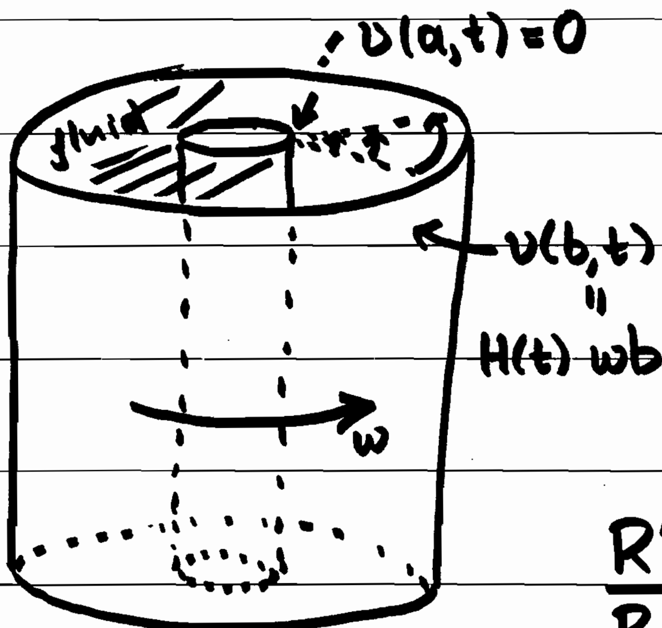
Ex. Between two concentric cylinders of infinite length there is a viscous fluid. At $t=0$, outer cylinder begins rotating. Determine motion of liquid (assumed laminar)

$$v_t = k\left(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v\right), \quad a<r<b, \; t>0$$

$$v(a,t) = 0$$

$$v(b,t) = H(t)\,\omega b \begin{cases} 0, & t<0 \\ \omega b, & t>0 \end{cases}$$

$$v(r,0) = 0 \qquad a \le r < b$$



Let $v(r,t) = R(r)\,T(t)$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} = \frac{T'}{kT} = -\lambda^2$$

$$R'' + \frac{1}{r}R' + \left(\lambda^2 - \frac{1}{r^2}\right)R = 0$$

$$R(r) = AJ_1(\lambda r) + BY_1(\lambda r)$$

$$\left.\begin{array}{l} R(a) = 0 \\ R(b) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} AJ_1(\lambda a) + BY_1(\lambda a) = 0 \\ AJ_1(\lambda b) + BY_1(\lambda b) = 0 \end{array}$$

Eigenvalues λ_n given by

$$\begin{vmatrix} J_1(\lambda_n a) & Y_1(\lambda_n a) \\ J_1(\lambda_n b) & Y_1(\lambda_n b) \end{vmatrix} = 0$$

Solutions: $R_n(r) = J_1(\lambda_n a)\, Y_1(\lambda_n r) - Y_1(\lambda_n a)\, J_1(\lambda_n r)$

$$T' + \lambda_n^2 k T = 0 \Rightarrow T_n(t) = e^{-\lambda_n^2 k t}$$

But if we set $v(r,t) = \sum_1^\infty b_n e^{-\lambda_n^2 kt} R_n(r)$, then

$v(b,t) = 0$ since $R_n(b)$ 0.

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Assume $v(r,t) = \sum_{n=1}^{\infty} a_n(t) R_n(r)$

where

$$a_n(t) = \frac{1}{N_n} \int_a^b \overset{\text{weight}}{r}\, v(r,t)\, R_n(r)\, dr$$

$$\left(R_n \text{ orthogonal}; \quad N_n = \int_a^b R_n^2(r) \underbrace{r}_{\text{weight for Bessel}} dr \right)$$

Multiply (equ.) by $r R_n(r)$ and integrate w.r.t. r, $\underline{a \text{ to } b}$ to obtain

$$\int_a^b v_t R_n r\, dr = k \int_a^b v_r R_n\, dr + k \int_a^b v_{rr} R_n r\, dr$$

$$- k \int_a^b \frac{1}{r} v R_n\, dr$$

$$\Rightarrow N_n a_n' = - \underbrace{\cancel{k v_r R_n r \Big|_a^b}}_{=0} - k \int_a^b v_r (r R_n)' dr + \underbrace{\cancel{k v R_n \Big|_a^b}}_{=0} -$$

$\longrightarrow$ over

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$$\cdots \quad -k\int_a^b v\, R_n'\, dr - k\int_a^b \frac{1}{r} v\, R_n\, dr$$

$$= -k(rR_n)'v\Big|_a^b + k\int_a^b (rR_n)''v\, dr - k\int_a^b v\, R_n'\, dr - k\int_a^b \frac{1}{r} v\, R_n\, dr$$

$$= -k(rR_n' + R_n)v\Big|_a^b + k\int_a^b \left[(rR_n)'' - R_n' - \frac{1}{r}R_n\right] v\, dr$$

$$= -kb\, R_n'(b)\, \omega b + k\int_a^b \left[rR_n'' + 2R_n' - R_n' - \frac{1}{r}R_n\right] v\, dr$$

$$= -k\omega b^2 R_n'(b) + k\int_a^b \underbrace{\left(R_n'' + \frac{1}{r}R_n' - \frac{1}{r^2}R_n\right)}_{= -\lambda_n^2 R_n} v\, r\, dr$$

$$= -k\omega b^2 R_n'(b) - k\lambda_n^2 \underbrace{\int_a^b R_n\, v\, r\, dr}_{= N_n a_n}$$

$$\therefore \quad a_n' + \lambda_n^2 k\, a_n = \frac{-k\omega b^2 R_n'(b)}{N_n} \quad ; \quad a_n(0) = 0$$

So $a_n(t) = \frac{k\omega b^2 R_n'(b)}{kN_n\lambda_n^2}\left(-1+e^{-\lambda_n^2 kt}\right)$

Convergence at $r=b$ non-uniform

Alternatively: $\begin{cases} u_t = k\left(u_{rr}+\frac{1}{r}u_r-\frac{1}{r^2}u\right) \\ u(a,t)=0\,,\quad u(b,t)=0 \\ u(r,0) = -w(r,0) \end{cases}$

with $w(r,t)$ a particular solution and $u(r,t)=v(r,t)-w(r,t)$

By inspection, let $w(r,t)=\omega r$; then $u(r,t)=\sum_1^\infty b_n e^{-\lambda_n^2 kt} R_n(r)$

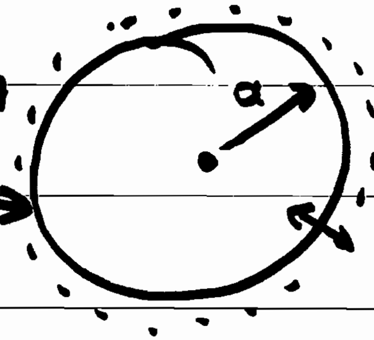
works if $u(r,0) = -\omega r = \sum_1^\infty b_n R_n(r)$

$$b_n = -\frac{\omega}{N_n}\int_a^b r\, R_n(r)\, r\, dr \qquad \text{(same answer)}$$

But 1^{st} approach preferable for more complicated motions, eg. if $v(b,t)=F(t)$.

Ex. $P_{rr} + \frac{1}{r} P_r + \frac{1}{r^2} P_{\theta\theta} = \frac{1}{c^2} P_{tt}$

$P(a,\theta,t) = f(\theta)\, e^{-i\omega t}$



$P = e^{-i\omega t}\, p(r,\theta)$

$p_{rr} + \frac{1}{r} p_r + \frac{1}{r^2} p_{\theta\theta} = -\left(\frac{\omega}{c}\right)^2 p \quad ; \quad p = R\Theta$

$r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \left(\frac{\omega}{c}\right)^2\right) = -\frac{\Theta''}{\Theta} = k^2 \quad ; \quad \Theta(\theta) = A_k \cos k\theta + B_k \sin k\theta$

$R'' + \frac{1}{r} R' + \left[\left(\frac{\omega}{c}\right)^2 - k^2/r^2\right] R = 0 \; : \; R_k(r) = Y_k\left(\frac{\omega}{c} r\right), J_k\left(\frac{\omega}{c} r\right)$

At $r = \infty$: want outgoing waves; write using

Hankel functions $H_k^{(1)}(x) = J_k(x) + i Y_k(x)$, $H_k^{(2)}(x) = H_k^{(1)*}(x)$

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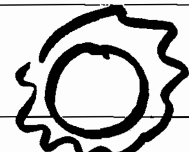
outgoing waves



$p = f(\theta)e^{-i\omega t}$ $\qquad p_k \sim \sqrt{\frac{2}{\pi x}}\, e^{-i\omega t \pm i\left(r - \frac{k\pi}{2} - \frac{\pi}{4}\right)}$

$$H_k^{(1)}(x) \simeq \sqrt{\frac{2}{\pi x}}\, e^{i\left(x - \frac{k\pi}{2} - \frac{\pi}{4}\right)}$$

Since $(-i\omega t + ir)$ is outgoing, keep only $H_k^{(1)}(x)$. So

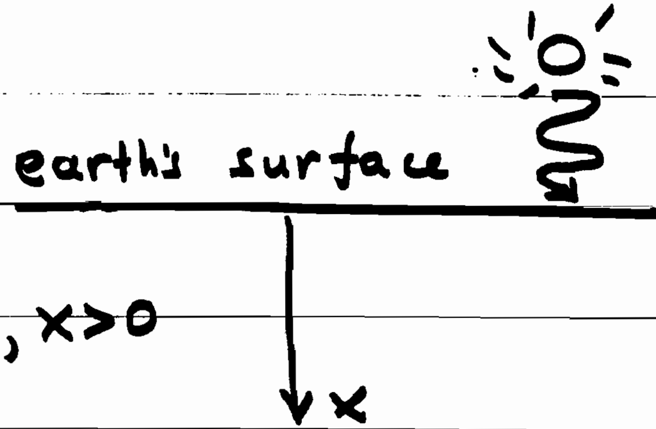
$$p(r,\theta,t) = e^{-i\omega t}\left\{\sum_{k=1}^{\infty}\left(A_k \cos k\theta + B_k \sin k\theta\right) H_k^{(1)}\left(\frac{\omega}{c} r\right) + \frac{a_0}{2} H_0^{(1)}\right\}$$

At $r = a$, determine A_k, B_k from $f(\theta)$.

As $r \to \infty$, this becomes $\qquad \left(A_k = a_k / H_k^{(1)}\left(\frac{\omega}{c} a\right) \text{ etc.}\right)$

$$p(r,\theta,t) \underset{r\to\infty}{\sim} e^{-i\omega(t - r/c)} \sum_{k=1}^{\infty}\left(A_k \cos k\theta + B_k \sin k\theta\right) e^{-i\frac{\pi}{2}\left(k + \frac{1}{2}\right)} + \dots$$

Problem:



$$\frac{\partial T}{\partial x} = \alpha^2 \frac{\partial^2 T}{\partial x^2}, \quad x>0$$

$$T(0,t) = T_0 \cos\omega t$$

Find temperature as function of depth.

Consider $u_t = \alpha^2 u_{xx}$; $u(0,t) = T_0 e^{\pm i\omega t}$

$\Rightarrow u(x,t) = v(x)\, e^{\pm i\omega t}$ $\quad (T(x,t) = \mathrm{Re}[u])$

Then $\pm i\omega v = \alpha^2 v'' \Rightarrow v'' \mp \frac{i\omega}{\alpha^2} v = 0$

let $v(x) = e^{\lambda x}$: $\lambda^2 \pm \frac{i\omega}{\alpha^2} = 0 \Rightarrow \lambda = \pm \frac{e^{i\pi/4}\sqrt{\omega}}{\alpha}$

$\downarrow$ two roots for $\pm i\omega$

$$\Rightarrow v(x) = e^{\pm \frac{\sqrt{\omega}}{\alpha}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)x}$$

Solutions of form

$$e^{i\omega t}\, e^{\pm \frac{\sqrt{\omega}}{\alpha}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)x}$$

(or $e^{-i\omega t}\, e^{\pm}\ldots$).

For propagation into the earth:

(and attenuation as $x \to \infty$):

$$u(x,t) = T_0\, e^{i\omega t}\, e^{-\frac{\sqrt{\omega}}{\alpha}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)x}$$

$$= T_0\, e^{-\frac{1}{\alpha}\sqrt{\frac{\omega}{2}}\,x}\, e^{i\left(\omega t - \frac{1}{\alpha}\sqrt{\frac{\omega}{2}}\,x\right)}$$

Then $T(x,t) = \mathrm{Re}(u) = T_0\, e^{-\frac{1}{\alpha}\sqrt{\frac{\omega}{2}}\,x} \cos\left(\omega t - \sqrt{\frac{\omega}{2}}\frac{x}{\alpha}\right)$

At $x = x_0$, fluctuation between $\pm T_0\, e^{-\frac{1}{\alpha}\sqrt{\frac{\omega}{2}}\,x_0}$,

time lag $\sqrt{\frac{\omega}{2}}\frac{x_0}{\alpha}$.