EIGENFUNCTION EXPANSIONS

Sturm-Liouville theory

$\{\phi_n(x)\}$, $x \in [a,b]$ $\quad \langle \phi_n, \phi_m \rangle = \int_a^b \phi_n(x)\bar{\phi}_m(x)\,dx = \delta_{nm} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$ (1)

orthonormal set:

let $f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$: (2) $\quad \int_a^b f(x)\,\bar{\phi}_m(x)\,dx = \sum_0^{\infty} c_n \delta_{nm} = c_m$ (3)

<u>Delta function</u>: $\quad \delta(x-\xi) = \sum_0^{\infty} \phi_n(x)\,\bar{\phi}(\xi)$ (4)

$$\left(\int_a^b f(\xi)\delta(x-\xi)\,d\xi = f(x) = \sum_0^{\infty} \phi_n(x) \int_a^b f(\xi)\bar{\phi}_m(\xi)\,d\xi \right)$$

<u>Examples</u>:

$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx \quad : \quad \frac{2}{\pi}\int_0^{\pi} \sin nx \sin mx\,dx = \delta_{nm}$ (1)′

$0 \leq x \leq \pi$

(Fourier sine series)

$$\begin{cases} f(x) = \sqrt{\frac{2}{\pi}} \sum_1^{\infty} c_n \sin nx & (2)' \\ c_n = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x)\sin nx\,dx & (3)' \end{cases}$$

Fourier cosine series:

$\phi_n(x) = \begin{cases} 1/\sqrt{\pi}, & n = 0 \\ \sqrt{\frac{2}{\pi}}\cos nx \end{cases}$;

$0 \leq x \leq \pi$

Complex Fourier:

$\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad -\pi \leq x \leq \pi$

Legendre polynomials: $\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{2}{2n+1}\,\delta_{nm}$

let $\phi_n(x) = \sqrt{\frac{2n+1}{2}}\,P_n(x)$: o.n. set on $-1 \le x \le 1$

$\left(f(x) = \sum_0^\infty a_n P_n(x) \right.$; $\left. a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\,P_n(x)\,dx \right)$

↑ Examples of o.n. families with <u>countably many</u> members.

Can extend to families with <u>uncountably</u> many members:

Fourier transform: $\phi(k,x) = \frac{1}{\sqrt{2\pi}}\, e^{ikx}$, $-\infty < k, x < \infty$

$$\left.\begin{aligned} &(5)\quad f(x) = \int_{-\infty}^{\infty} F(k)\,e^{ikx}\,dx \\ &(6)\quad F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx \end{aligned}\right\} \longrightarrow \left.\begin{aligned} f(x) &= \int_{-\infty}^{\infty} G(k)\,\phi(k,x)\,dk \\ G(k) &= \int_{-\infty}^{\infty} f(x)\,\bar{\phi}(k,x)\,dx \end{aligned}\right\} \begin{aligned}&(7)\\&(8)\end{aligned}$$

$$\left(G(k) = \sqrt{2\pi}\, F(k) \right)$$

Analogies: since $\delta(k-\bar{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(k-\bar{k})x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-\bar{k})}\,dx$

$$\Rightarrow \delta(k-\bar{k}) = \int_{-\infty}^{\infty} \phi(k,x)\,\bar{\phi}(\bar{k},x)\,dx \qquad (9)$$

Continuous

$$\delta(k-K) = \int_{-\infty}^{\infty} \phi(k,x)\,\bar{\phi}(K,x)\,dx \quad \longleftrightarrow \quad \delta_{nm} = \int_a^b \phi_n(x)\,\bar{\phi}_m(x)\,dx \qquad (9)$$

$$\delta(x-\xi) = \int_{-\infty}^{\infty} \phi(k,x)\,\bar{\phi}(k,\xi)\,dk \quad \longleftrightarrow \quad \delta(x-\xi) = {\sum_0^{\infty}}' \phi_n(x)\,\bar{\phi}_n(\xi) \qquad (10)$$

Other o.n. families of continuous type:

$$\phi(k,x) = \sqrt{\frac{2}{\pi}}\,\sin kx\,, \quad 0 \le x < \infty \quad \text{(sine)}$$

$$\phi(k,x) = \sqrt{\frac{2}{\pi}}\,\cos kx\,, \quad 0 \le x < \infty \quad \text{(cosine)}$$

Hankel transform:

$$\left.\begin{aligned} \phi(k,r) &= \sqrt{kr}\,J_0(kr) \\ F(k) &= 2\pi\sqrt{k}\,H(k) \\ f(r) &= \sqrt{r}\,h(r) \end{aligned}\right\} \qquad \begin{aligned} F(k) &= \int_0^{\infty} f(r)\,\phi(k,r)\,dr \\ f(r) &= \int_0^{\infty} F(k)\,\phi(k,r)\,dk \end{aligned}$$

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Eigenvalue problems

Ex: $y'' + \lambda y = 0 \quad , \; y(0) = y(\pi) = 0 \qquad (11)$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$y(x) = \sin\sqrt{\lambda}\, x \qquad\qquad \sqrt{\lambda}\,\pi = n\pi \Rightarrow \lambda = n^2$$

Eigenvalues: $\lambda_n = n^2$ Eigenfunctions $y_n = \sin nx$

o.n. set: $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ (sine series).

Sturm-Liouville system: $L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x) \qquad a \le x \le b$

let $u(a) = v(a) = 0$, $u(b) = v(b) = 0$. Then

$$\langle v, Lu\rangle - \langle Lv, u\rangle = \int_a^b \left\{ v\frac{d}{dx}\left(p\frac{du}{dx}\right) - u\frac{d}{dx}\left(p\frac{dv}{dx}\right)\right\} dx$$

(assume u, v real)

$$= \left(p\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\right|_a^b = 0 \qquad (12)$$

L : self-adjoint differential operator $\left(\begin{array}{l}\text{B.C.:} \\ y(a) = y(b) = 0\end{array}\right)$

S-L system (regular)

$$\begin{cases} L(y) + \lambda \rho y = 0, \quad y(a) = y(b) = 0 & (13) \\ -\infty < a < b < \infty; \quad p(x), \rho(x) \neq 0 \text{ in } [a,b] \end{cases}$$

Theorem: there are infinitely many eigenvalues $\lambda = \lambda_n$ for which (13) has solution $y = u_n(x)$. This sequence of λ_n's has no accumulation point except $\lambda = \infty$. The eigenfunctions $u_n(x)$ are orthonormal with weighting $\rho(x)$; (*1) furthermore, the $u_n(x)$ are complete (*2)

(*1) $\int_a^b \rho(x) u_n(x) u_m(x)\, dx = 0, \; n \neq m$. Define $N_n = \left(\int_a^b \rho(x) u_n^2 \, dx \right)^{1/2}$

Let $\phi_n(x) = \sqrt{\rho(x)}\, u_n(x) / N_n$; $\phi_n(x)$ orthonormal.

(*2) If $f(x)$ smooth in $[a,b]$: $f(x) = \sum_0^\infty c_n u_n(x)$

$$c_n = \frac{1}{N_n^2} \int_a^b \rho(x) f(x) u_n(x)\, dx$$

Ex: $L = \frac{d}{dx}\left(x \frac{d}{dx}\right)$; $\rho(x) = \frac{1}{x}$, $a = 1$, $b = 2$.

S-L system: $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \lambda y = 0$; $y(1) = y(2) = 0$ (14)

gen. solution $y(x) = A \sin(\sqrt{\lambda} \ln x) + B \cos(\sqrt{\lambda} \ln x)$

$y(1) = 0 \Rightarrow B = 0$; $y(2) = 0 \Rightarrow \sin(\sqrt{\lambda} \ln 2) = 0$

$\Rightarrow \sqrt{\lambda} \ln 2 = n\pi \Rightarrow \lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2$

Eigenfunctions: $u_n(x) = \sin\left(\frac{n\pi}{\ln 2} \ln x\right)$

(i) o.n. set: $\phi_n(x) = \frac{1}{N_n \sqrt{x}} u_n(x)$, $N_n = \left(\int_1^2 \frac{u_n^2(x)}{x} dx\right)^{1/2}$

(ii) $f(x)$ smooth on $1 \leq x \leq 2$: $f(x) = \sum_{n=1}^{\infty} c_n u_n(x)$

$c_n = \frac{1}{N_n^2} \int_a^b \frac{f(x) u_n(x)}{x} dx$

Singular S-L system : a or b becomes infinite
p(x) or $\rho(x)$ become 0 or ∞ on $[a,b]$

Results of theorem still hold provided all eigenfunctions are square integrable : $\int_a^b \rho(x)\, u_n^2(x)\, dx \equiv N_n^2 < \infty$.

Ex: $\frac{d}{dx}\left[\underset{\overset{\shortparallel}{0}}{(1-x^2)}\frac{dy}{dx}\right] + \lambda y = 0$; $y(-1), y(1) < \infty$
0 at a (=−1), b (=1) (Legendre)

$\lambda_n = n(n+1)$; $u_n = P_n(x)$

Ex: $\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \lambda x y = 0$; $y(0) < \infty$, $y(1) = 0$
$\lambda_n = j_n^2$ $(p(0) = \rho(0) = 0)$

j_n: n-th zero of $J_0(x)$

$u_n(x) = J_0(j_n x)$: $\int_0^1 x\, u_n(x)\, u_m(x)\, dx = 0$, $n \neq m$.

Fourier-Bessel series: $f(x) = \sum_0^\infty c_n J_0(j_n x)$; $c_n = \int_0^1 x f J_0(j_n x)\, dx / N_n^2$

In S-L systems, self-adjointness $\Rightarrow$ orthogonality

$$Lu_n + \lambda_n \rho u_n = 0 \quad ; \qquad \langle u, v \rangle = \int_a^b u \bar{v}\, dx$$

$$\langle u_m, Lu_n \rangle = -\langle u_m, \lambda_n \rho u_n \rangle = -\bar{\lambda}_n \langle u_m, \rho u_n \rangle \tag{15}$$

$$\langle Lu_m, u_n \rangle = -\langle \lambda_m \rho u_m, u_n \rangle = -\lambda_m \langle \rho u_m, u_n \rangle = -\lambda_m \langle u_m, \rho u_n \rangle \tag{16}$$

$(\rho(x)$ real$)$

$$\Rightarrow \underbrace{\langle u_m, Lu_n \rangle - \langle Lu_m, u_n \rangle}_{=0 \text{ (self-adjoint)}} = (\lambda_m - \bar{\lambda}_n) \underbrace{\langle u_m, \rho u_n \rangle}_{=0} \tag{17}$$

(self-adjoint) $\Rightarrow$

If $m = n$: $(\lambda_n - \bar{\lambda}_n) \int_a^b \rho\, u_n \bar{u}_n\, dx = 0$; but $\int_a^b \rho |u(x)|^2 dx \neq 0$

$\Rightarrow \lambda_n = \bar{\lambda}_n$: eigenvalues of S-L real.

(By linearity, can show that can always take $u_n(x)$ to be real)

Degenerate eigenvalue: can have two (or more) eigenfunctions (can still find orthonormal basis).

Ex: $y'' + \lambda y = 0 \,;\quad y(0) = y(2\pi),\ y'(0) = y'(2\pi)$ (20)

Self-adjoint: $\langle u, v'' \rangle - \langle u'', v \rangle = 0$ $\left(\substack{u,v \text{ sat.} \\ \text{B.C.}} \right)$

$\Rightarrow \lambda = \lambda_n = n^2 \,;\ u_n(x) = \cos nx\,,\ v_n(x) = \sin nx$

Sturm's 1st Comparison theorem:

$$\left.\begin{aligned} \frac{d}{dx}\left(p_1 \frac{du}{dx}\right) + g_1 u = 0 \\ \frac{d}{dx}\left(p_2 \frac{dv}{dx}\right) + g_2 v = 0 \end{aligned}\right\} \quad \begin{aligned} & g_1 \le g_2 \\ & p > 0 \\ & a \le x \le b \end{aligned}$$

(if $0 \le p_2 \le p_1$, also true).

$\Rightarrow$ there is at least one root of $v(x)$ between any two roots of $u(x)$.

(also, if $u(a) = v(a) = 0$, then $v(x)$ vanishes before $u(x)$)

S-L: $\frac{d}{dx}\left(p\frac{dy}{dx}\right) + g\,y = 0$; $g(x) = \lambda \rho(x) - q(x)$, $\quad$ (21)

$\rho(x) > 0$

increase $\lambda \longrightarrow$ solution becomes more oscillatory

Claim: as $\lambda \to \infty$, distance bet. successive zeros $\to 0$

(Indeed: let u satisfy $\frac{d}{dx}\left(P\frac{du}{dx}\right) + (\lambda \rho_0 - Q)\,u = 0$ $\quad$ (22)

$P = \max\limits_{[a,b]} p(x)$, $\rho_0 = \min\limits_{[a,b]} \rho(x)$, $Q = \max\limits_{[a,b]} q(x)$

with $y(a) = u(a) = 0$.

Then between any two zeros of $u(x)$ there is a zero of $y(x)$. But solutions of (22) sinusoids, of period $T = 2\pi\sqrt{\frac{P}{\lambda\rho_0 - Q}} \longrightarrow 0$ as $\lambda \to \infty$)

$\Rightarrow$

Similarly, compare $y(x)$ with the solution of

$$\frac{d}{dx}\left(p_0 \frac{du}{dx}\right) + (\lambda \mathcal{P} - q_0)u = 0 \qquad (23)$$

$$u(a) = 0$$

$\Rightarrow$ between any two zeros of $y(x)$ there is a zero of $u(x)$; therefore the <u>minimum</u> separation of the zeros of $y(x)$ is $\pi\sqrt{\dfrac{p_0}{\lambda \mathcal{P} - q_0}}$

$$\left.\begin{array}{l} p_0 = \min p(x),\ q_0 = \min q(x) \\ \mathcal{P} = \max \rho(x) \end{array}\right\}$$

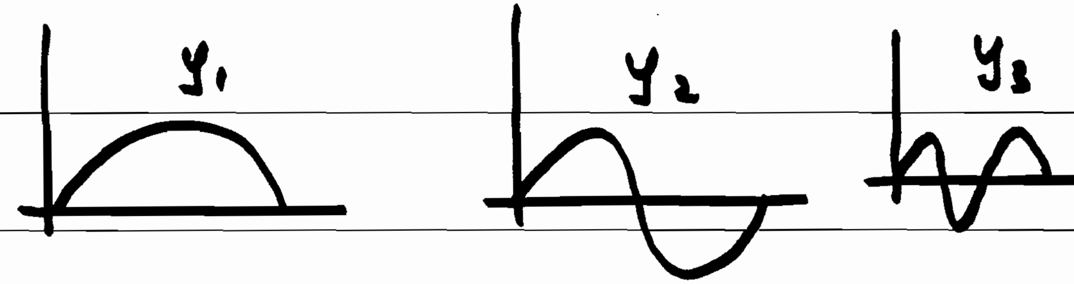
③ let y_n be n-th zero of $y(x)$:

$$n\pi\sqrt{\frac{p_0}{\lambda \mathcal{P} - q_0}} < x_n - a < n\pi\sqrt{\frac{P}{\lambda \rho_0 - Q}}$$

☺ If $y(a) = y(b) = 0$: $\begin{cases} y_1 \text{ has no zeros in } (a,b) \\ y_2 \text{ has 1 zero} \\ y_n \text{ has } (n-1) \text{ zeros} \end{cases}$

Ex: $y'' + \lambda y = 0$

$y(0) = y(2\pi) = 0$



Can show ($x_n = b$ if $\lambda = \lambda_n$):

$$\frac{1}{\mathcal{P}}\left\{\left(\frac{n\pi}{b-a}\right)^2 p_0 + q_0\right\} \le \lambda_n \le \frac{1}{\rho_0}\left\{\left(\frac{n\pi}{b-a}\right)^2 P + Q\right\}$$

☺ $\lambda_n \sim n^2$ for n large