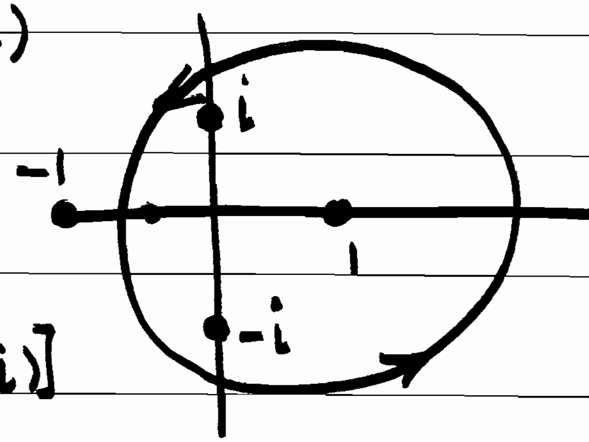
RESIDUE THEOREM

Ex. $I = \oint_c \frac{z\,dz}{(z^2-1)^2(z^2+1)}$



poles inside: $\pm i$, 1

$I = 2\pi i \sum \text{Res}$

$\pm i$: $\lim_{z \to \pm i} [f(z)\cdot(z \mp i)]$

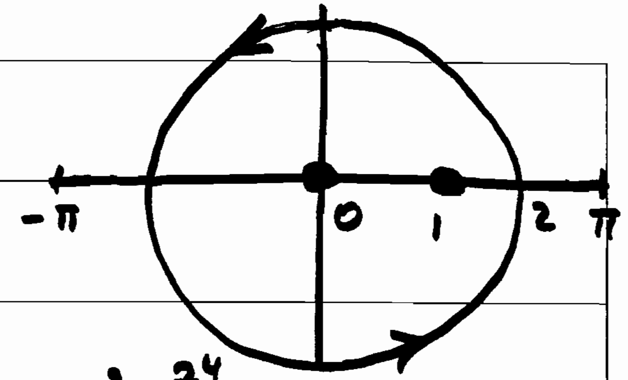
$= \lim_{z \to \pm i} \left[\frac{z}{(z^2-1)^2(z \pm i)}\right] = \frac{1}{8}$

1: (double): $\lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right] = \lim_{z \to 1} \frac{d}{dz}\left(\frac{z}{(z+1)^2(z^2+1)}\right) = \left.\frac{-3z^3-z^2-z+1}{(z+1)^3(z^2+1)}\right|_1 = -\frac{1}{8}$

$I = 2\pi i \left(\text{Res}(+i) + \text{Res}(-i) + \text{Res}(1)\right)$

$= 2\pi i \left(\frac{1}{8} + \frac{1}{8} - \frac{1}{8}\right) = \frac{\pi i}{4}$

Ex: $I = \oint_c \frac{\cot z}{z(z-1)}\,dz$



$\cot z = \frac{\cos z}{\sin z}$

$\sin z = z - \frac{z^3}{6} + \cdots \Rightarrow z \sin z = z^2 - \frac{z^4}{6} + \cdots$ double zero

$\Rightarrow$ poles $z=1$ (simple), $z=0$ (double)

$$\text{Res}(1) = \lim_{z\to 1} \frac{\cos z}{z \sin z} = \cot 1$$

$$\text{Res}(0) = \lim_{z\to 0} \frac{d}{dz} z^2 f(z) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z\cot z}{z-1}\right)$$

Now: $$\frac{d}{dz}\Big(\underbrace{\frac{\cos z}{z-1}}_{\text{easy}} \cdot \underbrace{\frac{z}{\sin z}}_{0/0}\Big) = \Big[\underbrace{\frac{-\sin z}{z-1}}_{0} - \underbrace{\frac{\cos z}{(z-1)^2}}_{-1}\Big]\underbrace{\frac{z}{\sin z}}_{1} + \underbrace{\frac{\cos z}{z-1}}_{-1}\left(\frac{1}{\sin z} - \frac{z\cos z}{\sin^2 z}\right)$$

$z \to 0$

while $$\lim_{z\to 0}\left(\frac{\sin z - z\cos z}{\sin^2 z}\right) = \lim_{z\to 0}\left(\frac{\cancel{\cos z} - \cancel{\cos z} + z\cancel{\sin z}}{2\cancel{\sin z}\cos z}\right) = \frac{0}{2} = 0$$

So $\text{Res}(0) = -1$; $\quad I = 2\pi i(\cot 1 - 1)$

DEFINITE INTEGRALS

Integrals of form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx, \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, \underset{\cos x + i\sin x}{\underset{\parallel}{e^{ix}}}\,dx$$

with P, Q polynomials; $\deg Q \geq \deg. P+2$

Also $\int_0^{\infty} f(x)\,dx$ if $f(x)$ is even, of form

Key ideas: triangle inequality $|a+b| \geq |a|-|b|$

$z = Re^{i\theta}$
$= R\cos\theta + iR\sin\theta$

$$|e^{iz}| = |e^{i(R\cos\theta + iR\sin\theta)}| = e^{-R\sin\theta}$$

since $|e^{iR\cos\theta}| = 1$

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}||e^{-y}| = e^{-y}$$

$$|e^{z}| = |e^{x+iy}| = |e^{x}||e^{-y}| = e^{x}$$

$$\left|\frac{1}{z^n+1}\right| = \left|\frac{1}{R^n e^{in\theta}+1}\right| \leq \frac{1}{R^n-1}$$

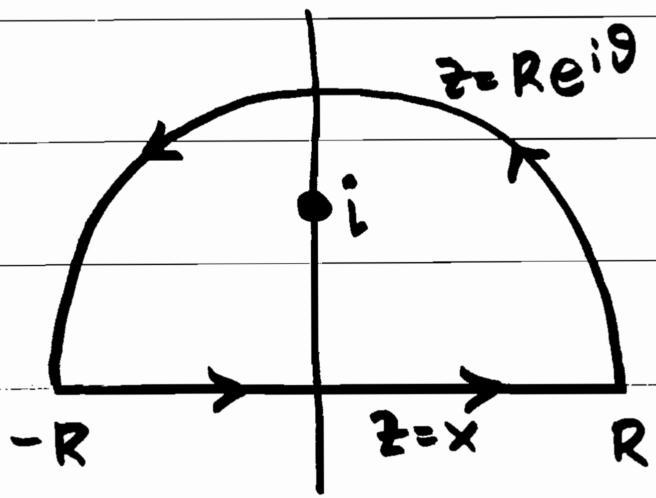
since $|R^n e^{in\theta}+1| \geq$
$\geq |R^n e^{in\theta}| - 1$
$= R^n - 1$

Ex. $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$; consider C as shown,

$$\oint_c \frac{dz}{1+z^2} = 2\pi i \cdot \mathrm{Res}(i) = 2\pi i \cdot \lim_{z\to i} (z-i)\frac{1}{z^2+1}$$

$$= 2\pi i \cdot \frac{1}{z+i}\Big|_{z=i} = \pi$$

But $\oint_c = \underbrace{\int_{-R}^{R} \frac{dx}{1+x^2}}_{= I_1(R)} + \underbrace{\int_{\theta=0}^{\pi} \frac{Rie^{i\theta}d\theta}{1+R^2e^{2i\theta}}}_{I_2(R)}$



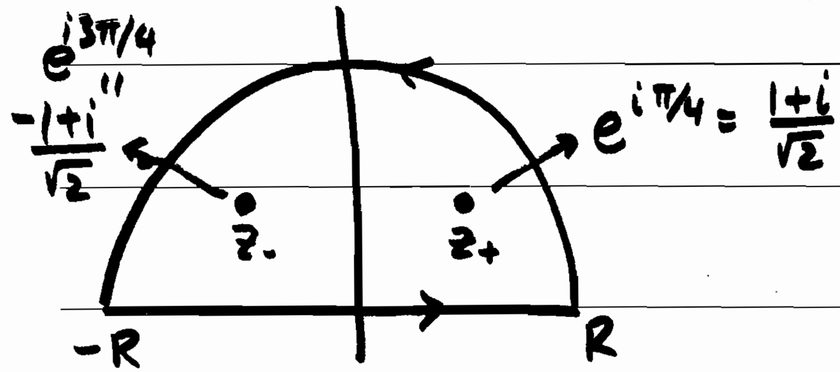
Now, this integral does not change as $R\to\infty$

So $\pi = \lim_{R\to\infty} \oint_c = \underbrace{\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2}}_{= I} + \lim_{R\to\infty} I_2(R)$

Now $|I_2(R)| \leq \int_{\theta=0}^{\pi} \frac{|Rie^{i\theta}|d\theta}{|1+R^2e^{2i\theta}|} \leq$

$$\leq \frac{R}{R^2-1}\int_0^{\pi} d\theta = \frac{\pi R}{R^2-1} \xrightarrow{R\to\infty} 0 \; ; \quad \boxed{I = \pi}$$

Ex $\displaystyle \int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{2\pi i}{2} \sum \text{Res}$



$$\text{Res}\left((\pm 1+i)/\sqrt{2}\right) = \lim_{z \to z_\pm} (z-z_\pm)/z^4 = \left.\frac{1}{4z^3}\right|_{z_\pm}$$

$$= \frac{1}{4z_\pm^3} = \left\{ \left.\frac{1}{4} e^{-i3\pi/4}\right._{z=e^{i\pi/4}} \text{ or } \left.\frac{1}{4} e^{-i9\pi/4}\right._{z=e^{i3\pi/4}} \right\}$$

$$I = \frac{2\pi i}{8}\left(e^{-i3\pi/4} + e^{-i9\pi/4}\right) = \frac{2\pi i}{8}\left(\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right) = \frac{\pi\sqrt{2}}{4}$$

Again, $\displaystyle \int_{-R}^{R} \to \int_{-\infty}^{\infty}$; $\displaystyle \int_0^\pi \to 0$ as $R \to \infty$

In general if

$$f(x) = \frac{P(x)}{Q(x)}, \text{ polynomials}$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \cdots \quad \text{while}$$

$$\oint_c \frac{P(z)}{Q(z)}\,dz = 2\pi i \cdot \sum \text{Res (upper half-plane)}$$

$$P(z) = p_0 z^m + \cdots,\ Q(z) = q_0 z^n$$

then

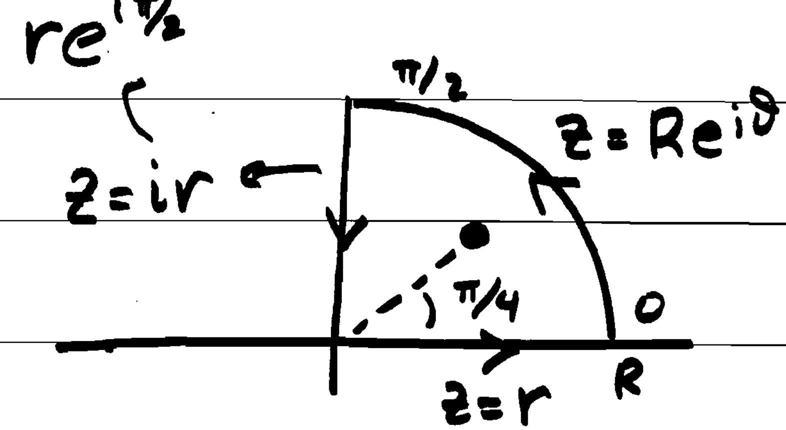
$$\left| \int_{\theta=0}^{\pi} \frac{(p_0 R^m e^{im\theta} + \cdots)\, Rie^{i\theta} d\theta}{q_0 R^n e^{in\theta} + \cdots} \right| \leq \frac{|p_0|}{|q_0|} \cdot \frac{(R^m + \cdots)R}{(R^n - \cdots)} \cdot \pi$$

which behaves like $\frac{R^{m+1}}{R^n}$ as $R \to \infty$, i.e. $\to 0$

if $\underline{n > m+1}$.

A. $R \to \infty$, contour will enclose all roots on u. per.

$$\int_0^\infty \frac{dx}{1+x^4}$$



$$\oint_C \frac{dz}{1+z^4} = 2\pi i \cdot \mathrm{Res}\,(e^{i\pi/4}) = 2\pi i \cdot \frac{1}{4e^{3i\pi/4}}$$

$$= \frac{2\pi i}{4\sqrt{2}}(-1-i)$$

But now $\displaystyle \oint = \int_0^R \frac{dr}{1+r^4} - \int_0^R \frac{i\,dr}{1+r^4} + \int_0^{\pi/2} \text{'''}\, d\theta$

$$= (1-i)\int_0^R \frac{dr}{1+r^4} + \int d\theta \xrightarrow{R\to\infty} (1-i)\int_0^\infty \frac{dx}{1+x^2}$$

$$\Rightarrow I = \frac{2\pi i}{4\sqrt{2}}(-1-i)\Big/(1-i) = \frac{\pi i}{2\sqrt{2}} \cdot -\frac{(1-1+2i)}{2} = \frac{\pi}{2\sqrt{2}} \text{ again.}$$

line $z = re^{i\pi/2}$: $z^4 = r^4 e^{i\cdot 4\pi/2} = r^4 e^{2\pi i} = r^4 \rightarrow z = e^{i2\pi/5}$

(Same idea works with any power: $z^5 : r^5 e^{i\cdot 10\pi/5}$

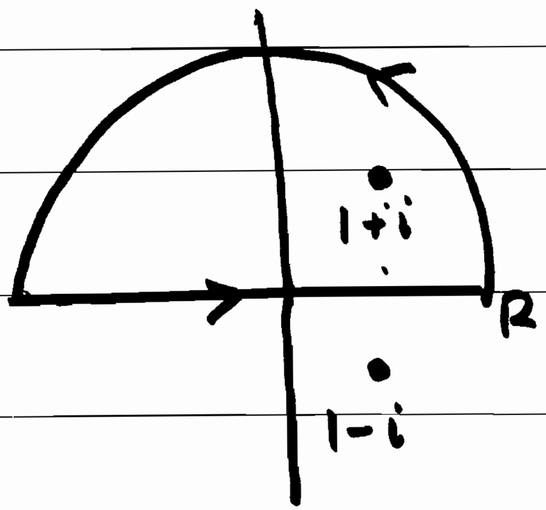
Sine-cosine transforms (or Fourier transforms)

$$I = \int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 - 2x + 2} \quad (\text{similar for } \sin x)$$

↳ consider $J = \int_{-\infty}^{\infty} \frac{e^{ix} dx}{x^2 - 2x + 2}$,

when finished, take real (or imag.) part.

$x^2 - 2x + 2 = (x-1)^2 + 1$: roots $1 \pm i$



$$\oint_R \frac{e^{iz}}{z^2 - 2z + 2} dz = 2\pi i \cdot \text{Res}(z = 1+i)$$

$$= 2\pi i \cdot \left. \frac{1}{z - 1 + i} \right|_{z = 1+i} = \frac{2\pi i}{2i}$$

$$= \pi$$

$$\int_{-R}^{R} \longrightarrow \int_{-\infty}^{\infty}$$

$$\oint_C = \int_{-R}^{R} + \int_0^{2\pi} \frac{e^{i(Re^{i\theta})}}{R^2 e^{2i\theta} - 2Re^{i\theta} + 2} Rie^{i\theta} d\theta$$

$$\left(e^{iRe^{i\theta}} = e^{iR\cos\theta - R\sin\theta} \right)$$

Detailed argument (although it fails in general case):

$$\left|\int_0^{2\pi} \cdots \right| \leq \int_0^{2\pi} \frac{R e^{-R\sin\vartheta}\, d\vartheta}{\underbrace{R^2 - 2R - 2}_{\text{make denom. as small as possible!}}}$$

$$\leq \int_0^{2\pi} \frac{R}{R^2-2R-2}\, d\vartheta = \frac{\pi R}{R^2-2R-2} \xrightarrow{R\to\infty} 0$$

here used $e^{-R\sin\vartheta} < 1$ since $0 \leq \vartheta \leq \pi \Rightarrow \sin\vartheta \geq 0$.

However, this argument is too crude, since as $R \to \infty$, $e^{-R\sin\vartheta} \to 0$ if $\cancel{\vartheta \neq 0, \pi}$. $\vartheta \in (0,\pi)$. Can we use this property to relax restriction that $\deg Q \geq \deg P + 2$ to $\deg Q \geq \deg P + 1$ for F...