466 '07-HOMEWORK 1

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due: Thursday, August 30, 2007

August 21, 2007

1. Is $u(x,y) = x\cos y\cosh x + y\sin y\sinh x$ the real part of an analytic function? Answer the same question for $u(x,y) = x\cos y\cosh x - y\sin y\sinh x$.

2. Derive the Cauchy-Riemann equations in polar coordinates and determine if $re^{\cos\theta}$ is a harmonic function.

3. If $f(z) = u(x,y) + iv(x,y)$, show that the curves $u(x,y) = \text{const}$ and $v(x,y) = \text{const}$ are orthogonal at every point where $f'(z)$ exists and is not zero. Sketch the families of curves for the case where $f(z) = u + iv = z^2$ and comment on the structure near the origin of the z-plane.

4. Establish the formula

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z},$$

for the sum of a finite geometric series. Then, derive the formulas

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}$$

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}$$

5. Write the following complex numbers in the form $a + ib$:

   (a) $(1 + 2i)^2$

   (b) $(1 + 2i)^9$

6. Find each of the roots and locate them geometrically

   (a) $\left(-2\sqrt{3} - 2i\right)^{1/4}$

   (b) $(-1 + i)^{1/3}$

7. Find all the values of z such that $z^5 = 32$.

8. Find all the values of i^i.

9. ($\tan^{-1} :=$ arctan)

   (a) Show that
$$\tan^{-1} z = k\pi + \frac{1}{2i}\mathrm{Log}\left(\frac{1 + iz}{1 - iz}\right)$$
   where $k = 0, \pm 1, \pm 2, \ldots$

   (b) Find the values of $\tan^{-1}(1 - 2i)$.

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1. $u(x,y)$ must be harmonic, and there exist $v(x,y)$ s.t. $u_x = v_y$, $u_y = -v_x$.

$u(x,y) = x\cos y\cosh x \pm y\sin y\sinh x$

(a) $u_x = \cos y\cosh x + x\cos y\sinh x \pm (y\sin y\cosh x) = v_{1,y}$

$\Rightarrow v_1 = \sin y\cosh x + x\sin y\sinh x \pm (-y\cos y + \sin y)\cosh x + f(x)$

since $\int^y y\sin y = -y\cos y + \sin y$

(b) $-u_y = x\sin y\cosh x \mp (\sin y\sinh x + y\cos y\sinh x) = v_{2,x}$

$\Rightarrow v_2 = (x\sinh x - \cosh x)\sin y \mp (\sin y\cosh x + y\cos y\cosh x) + g(y)$

Equating: $v_1 = v_2$ find $g(y) \equiv f(x) \equiv$ constant, C and equality only possible for lower sign. Then

$f(x,y) = (x\cos y\cosh x - y\sin y\sinh x) + i(x\sin y\sinh x + y\cos y\cosh x) + C$

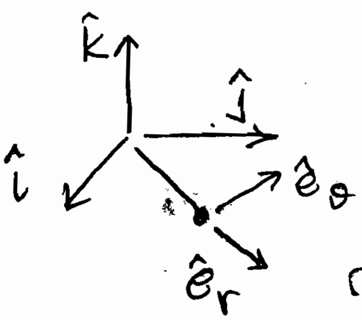
is analytic.

2. Given two harmonic conjugate functions $u(x,y)$, $v(x,y)$, the Cauchy-Riemann equations imply $\hat{k}\times\nabla u = \nabla v$.

Rewriting in polar coordinates, $\nabla u = \hat{e}_r\partial_r u + \frac{1}{r}\hat{e}_\theta\partial_\theta u$:

$\hat{k}\times\nabla u = \nabla v \Rightarrow \hat{k}\times(\hat{e}_r\partial_r u + \frac{1}{r}\hat{e}_\theta\partial_\theta u) = (\hat{e}_r\partial_r v + \frac{\hat{e}_\theta}{r}\partial_\theta v)$

$\Rightarrow$ (since $\hat{k}\times\hat{e}_r = \hat{e}_\theta$, $\hat{k}\times\hat{e}_\theta = -\hat{e}_r$)



$$-\frac{1}{r}\hat{e}_r\partial_\theta u + \hat{e}_\theta\partial_r u = \hat{e}_r\partial_r v + \frac{1}{r}\hat{e}_\theta\partial_\theta v$$

$$\Rightarrow \boxed{\partial_r u = \frac{1}{r}\partial_\theta v, \quad -\frac{1}{r}\partial_\theta u = \partial_r v}$$

Then $\partial_r(r\partial_r u) = \partial_{r\theta} v = -\partial_\theta\left(\frac{1}{r}\partial_\theta u\right)$

or $r\partial_r(r\partial_r u) + \partial_\theta^2 u = 0$ is Laplace eq.

over

4/4) The function $U = re^{\cos\theta}$ is not harmonic:

$$\partial_r U = e^{\cos\theta}, \quad r\partial_r(r\partial_r U) = r\partial_r(re^{\cos\theta}) = re^{\cos\theta} = U.$$

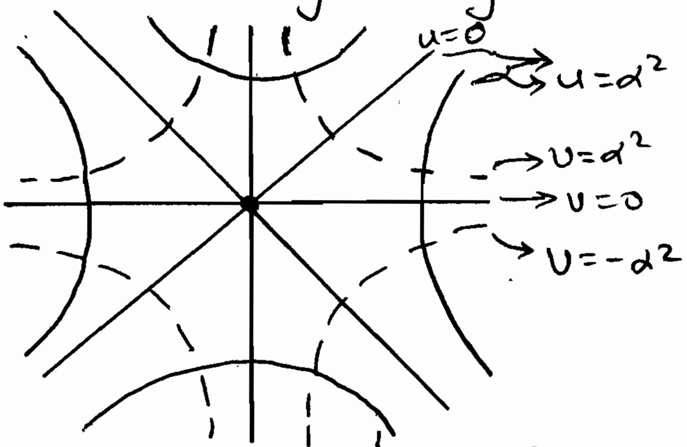
$$\text{But } \underline{\partial_\theta^2(re^{\cos\theta})} = \partial_\theta(-r\sin\theta e^{\cos\theta}) = \underline{r(\sin^2\theta - \cos\theta)e^{\cos\theta}}. \quad \neq$$

<u>3.</u> $u = \text{const} \Rightarrow \nabla u = 2(x\hat{i} - y\hat{j})$

$v = \text{const} \Rightarrow \nabla v = 2(y\hat{i} + x\hat{j})$

CR: $\hat{k} \times \nabla u = \nabla v \Rightarrow \nabla u \perp \nabla v$

$f = u + iv = x^2 - y^2 + i2xy$; level lines are families of hyperbolas



At $z=0$,
$\nabla u = \nabla v = 0$: no normal direction.

<u>4.</u> $(1 + z + \cdots + z^n)(1-z) = (1 + z + \cdots + z^n) - (z + \cdots + z^{n+1}) = 1 - z^{n+1}$

let $z = e^{i\theta} = \cos\theta + i\sin\theta$; $z^n = \cos n\theta + i\sin n\theta$

Then $\displaystyle\sum_0^n z^n = \underbrace{(1 + \cos\theta + \cdots + \cos n\theta)}_{\text{real part}} + i\underbrace{(\sin\theta + \cdots + \sin n\theta)}_{\text{imaginary part}}$

$$= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{e^{i(n+\frac{1}{2})\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}$$

$$= \left(\left[\cos(n+\tfrac{1}{2})\theta + i\sin(n+\tfrac{1}{2})\theta\right] - \left[\cos\theta/2 - i\sin\theta/2\right]\right) / 2i\sin\theta/2$$

$$= \underbrace{\frac{1}{2} + \frac{1}{2}\frac{\sin(n+\frac{1}{2})\theta}{\sin\theta/2}}_{\text{real part}} + i\underbrace{\left[\frac{1}{2}\cot\theta/2 - \frac{1}{2}\frac{\cos(n+\frac{1}{2})\theta}{\sin\theta/2}\right]}_{\text{imaginary part}}$$

s. 2/4)

5. (a) $(1+2i) = \sqrt{5}\left(\frac{1}{\sqrt{5}} + i\frac{2}{\sqrt{5}}\right)$; let $\vartheta = \tan^{-1} 2 \in (0, \frac{\pi}{2})$.

$\cos\vartheta = 1/\sqrt{5}$, $\sin\vartheta = 2/\sqrt{5}$; $\quad (1+2i)^k = 5^{k/2}(\cos k\vartheta + i\sin k\vartheta)$

$$\begin{cases} \cos 2\vartheta = \cos^2\vartheta - \sin^2\vartheta = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5} \\ \sin 2\vartheta = 2\cos\vartheta\sin\vartheta = 4/5 \end{cases}$$

$$(1+2i)^2 = 5\left(-\frac{3}{5} + i\frac{4}{5}\right)$$

$$(1+2i)^2 = -3+4i$$

$$\begin{cases} \cos 4\vartheta = \cos^2 2\vartheta - \sin^2 2\vartheta = -7/25 \\ \sin 4\vartheta = 2\cos 2\vartheta \sin 2\vartheta = -24/25 \end{cases}$$

$$\begin{cases} \cos 8\vartheta = \cos^2 4\vartheta - \sin^2 4\vartheta = -527/625 \\ \sin 8\vartheta = 2\cos 4\vartheta \sin 4\vartheta = 336/625 \end{cases}$$

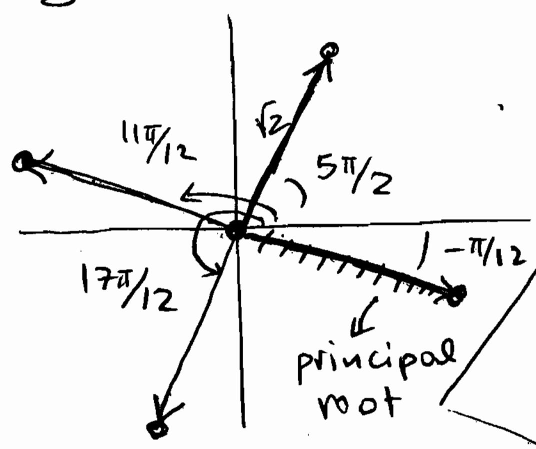
$$\cos 9\vartheta = \cos 8\vartheta \cos\vartheta - \sin 8\vartheta \sin\vartheta = -1199/625\sqrt{5}$$

$$\sin 9\vartheta = \sin 8\vartheta \cos\vartheta + \cos 8\vartheta \sin\vartheta = -718/625\sqrt{5}$$

(b) $\{(1+2i)^9 = -1199 - 718\,i\}$

6. $(2\sqrt{3} - 2i)^{1/4} = \left[4\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)\right]^{1/4} = \left[4e^{-i\pi/6 + 2k\pi i}\right]^{1/4} = \sqrt{2}\, e^{-i\frac{\pi}{12} + \frac{k\pi}{2}i}$

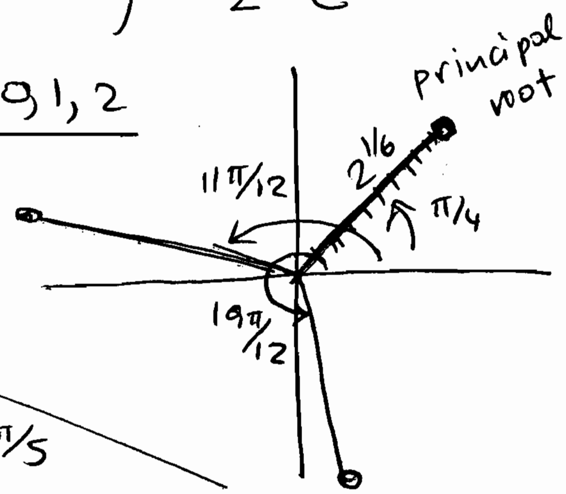
$$\text{Arg.} = \begin{cases} \xrightarrow{k=0} -\pi/12 \\ \xrightarrow{k=1} -\frac{\pi}{12} + \frac{\pi}{2} = \frac{5\pi}{12} \\ \xrightarrow{k=2} -\frac{\pi}{12} + \pi = \frac{11\pi}{12} \\ \xrightarrow{k=3} -\frac{\pi}{12} + \frac{3\pi}{2} = \frac{17\pi}{12} \end{cases}$$



(b) $(-1+i)^{1/3} = \left[\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]^{1/3}$

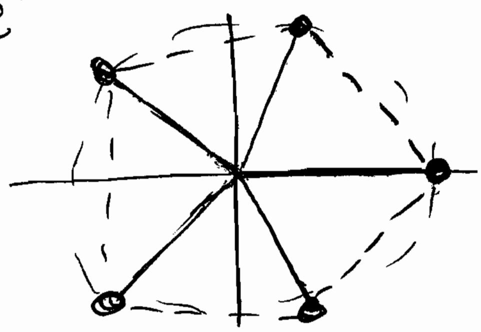
$$= 2^{1/6}\left(e^{i\frac{3\pi}{4} + 2k\pi i}\right)^{1/3} = 2^{1/6} e^{i\frac{\pi}{4} + \frac{2k\pi i}{3}}$$

$k = 0, 1, 2$



7. $z^5 = 32 = 2^5 e^{i\,2n\pi}$

$z = 2e^{i\,2n\pi/5} \longrightarrow 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, 8\pi/5$



8. $i^i = e^{i\pi/2 + 2n\pi i}$; $i^i = \left(e^{i\frac{\pi}{2}+2n\pi i}\right)^i = e^{-\frac{\pi}{2}-2n\pi}$

infinitely many values. $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

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9. Let $w = \tan^{-1} z \Rightarrow z = \tan w$

Then $z = \dfrac{\sin w}{\cos w} = \dfrac{\frac{1}{2i}(e^{iw} - e^{-iw})}{\frac{1}{2}(e^{iw} + e^{-iw})} = -i\,\dfrac{u - 1/u}{u + 1/u}$; $u = e^{iw}$

$$\Rightarrow iz = \frac{u^2 - 1}{u^2 + 1} \Rightarrow u^2 = \boxed{\frac{1+iz}{1-iz} = e^{2iw}}$$

$$\Rightarrow 2iw = \ln\left(\frac{1+iz}{1-iz}\right) = \mathrm{Log}\left(\frac{1+iz}{1-iz}\right) + 2n\pi i$$

$$\therefore \boxed{w = \frac{1}{2i}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right) + n\pi} \qquad \text{QED}$$

For $z = 1 - 2i$, $\dfrac{1+iz}{1-iz} = \dfrac{1 + i(1-2i)}{1 - i(1-2i)} = \dfrac{3+i}{-1-i}$

$$= \frac{(3+i)(-1+i)}{(-1-i)(-1+i)} = -2 + i = \sqrt{5}\left(-\frac{2}{\sqrt{5}} + i\frac{1}{\sqrt{5}}\right)$$

$$= re^{i\theta}$$

$$\mathrm{Log}\left(\frac{1+iz}{1-iz}\right) = \ln\sqrt{5} + i\theta = \frac{1}{2}\ln 5 + i\theta$$

$$\therefore \tan^{-1}(1-2i) = \frac{\theta}{2} - \frac{i}{4}\ln 5 + n\pi; \quad \theta = \tan^{-1}(-1/2)$$

$$\frac{\theta}{2} - \frac{i}{4}\ln 5 + n\pi = -13.3 - .4i + n\pi \qquad \theta = -26.6$$