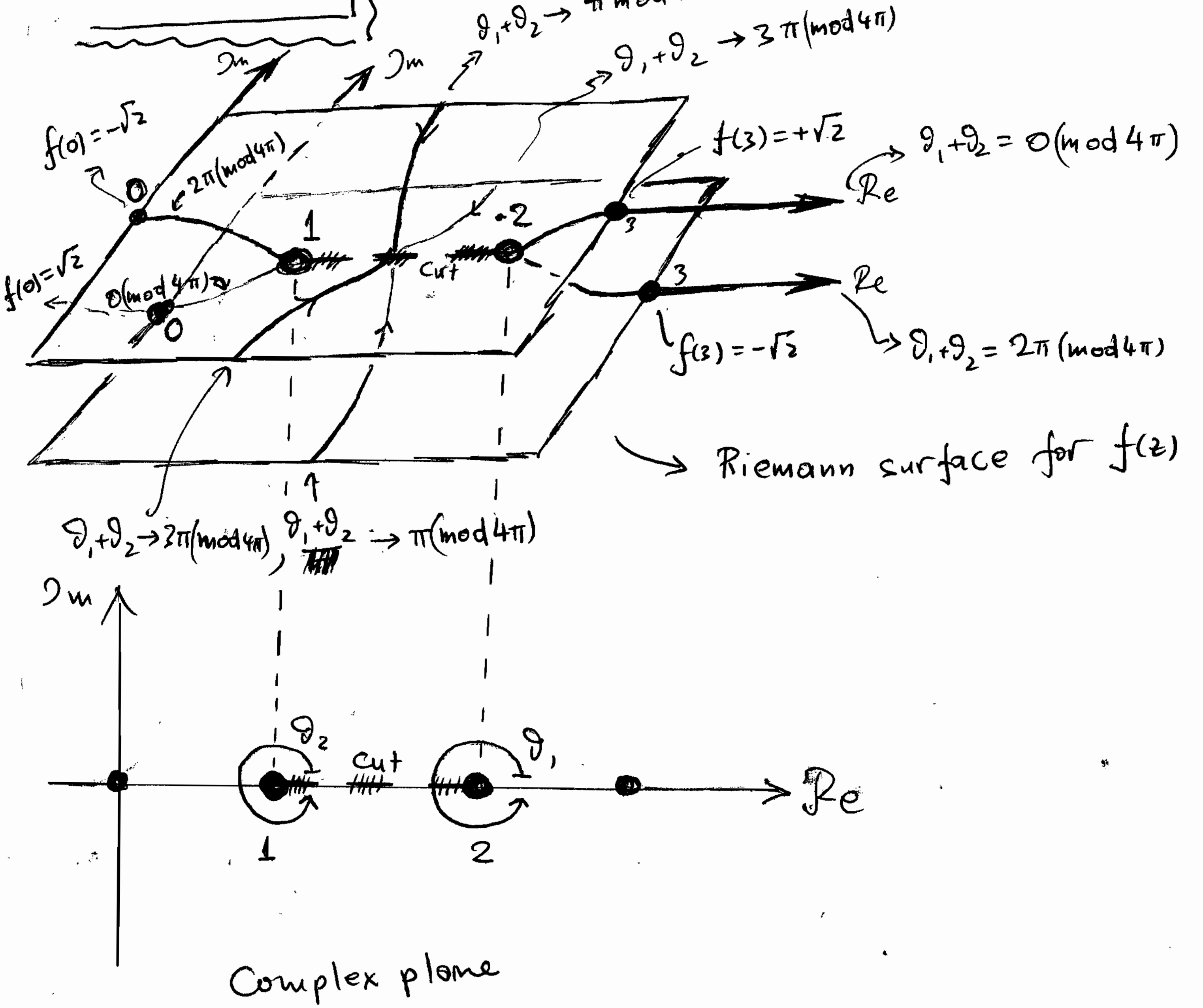
Solutions Set IV

(1)(a) $f(z) = \sqrt{(z-1)(z-2)}$



$f(z)$ has two branch points, at $z=1, 2$. Since in order to have a single valued function we need cuts starting at each branch point (here we can get away just by a single cut from $z=1$ to $z=2$), where the single-valued branch of $f(z)$ is discontinuous these are non-isolated singular points (check also that $f'(z)$ does not exist at $z=1, 2$).

For $z=\infty$: let $\zeta = 1/z$; $f(1/\zeta) \equiv g(\zeta)$

with $g(\zeta) = \frac{1}{\zeta}\sqrt{(1-\zeta)(1-2\zeta)}$. Since $(1-\zeta)^{1/2}, (1-2\zeta)^{1/2}$ are analytic at $\zeta=0$, we are left with a simple pole there, due to the "$1/\zeta$" factor.

Indeed, we can also see that from the Laurent series.

$$(1-\zeta)^{1/2} = \underset{(|\zeta|<1)}{1-\tfrac{1}{2}\zeta+\cdots}\ ;\quad (1-2\zeta)^{1/2} = \underset{(|\zeta|<1/2)}{1-\zeta+\cdots}$$

(binomial expansions, valid near $\zeta=0$)

$$\Rightarrow g(\zeta) = \frac{1}{\zeta}\left(1-\tfrac{1}{2}\zeta+\cdots\right)\left(1-\zeta+\cdots\right) = \frac{1}{\zeta} - \frac{3}{2} + \cdots$$

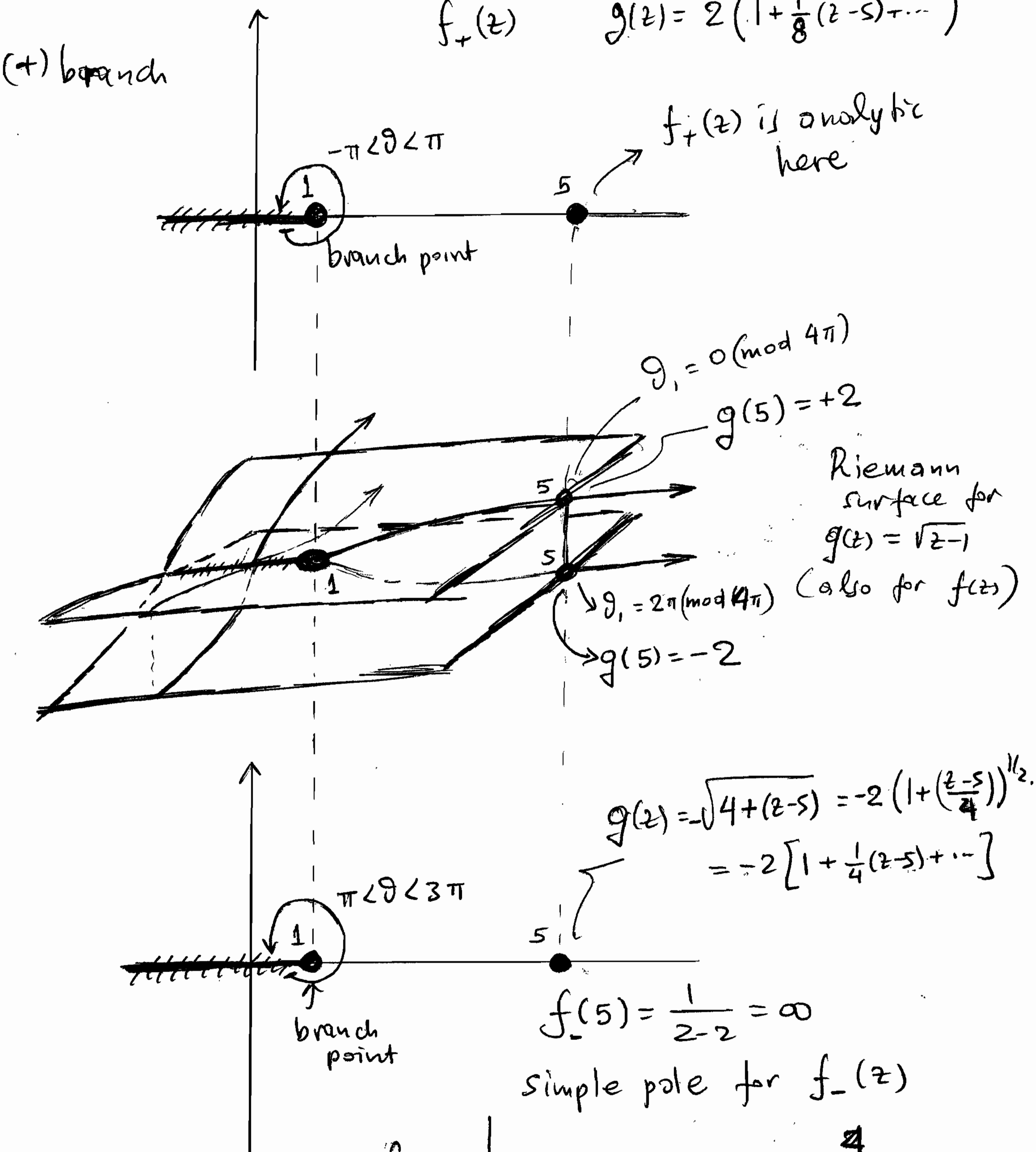
i.e. the Laurent series has only the neg. power (-1).

(b) $f(z) = (\cos z)^{-1}$; $\cos z$ is entire, so only finite singularities of $f(z)$ are found at zeroes of $\cos z$, i.e. at $z_n = (n+\frac{1}{2})\pi$, $n=0,\pm1,\pm2,\ldots$. Since $\cos z$ only has simple zeros there ($-\sin z_n = \cos' z_n \neq 0$), z_n are simple poles of $f(z)$

For $z=\infty$: since, for any R we can find N such that if $n>N$, $|z_n|>R$, there are infinitely many singular points of f (poles) in every neighborhood of ∞, so ∞ is nonisolated.

(1)(c) $\dfrac{1}{2+\sqrt{z-1}} = f(z)$; let $g(z) = \sqrt{z-1}$, then

to visualize what happens consider:



(−) branch $\quad f = \dfrac{1}{2-2\left(1+\frac{1}{8}(z-5)+\cdots\right)} = -\dfrac{4}{(z-5)+\cdots}$

For $z=\infty$: $h(\zeta) = f(\frac{1}{\zeta}) = \frac{1}{2+\sqrt{\frac{1}{\zeta}-1}}$

$$= \frac{\sqrt{\zeta}}{2\sqrt{\zeta}+\sqrt{1-\zeta}}$$

~~Although~~ $\zeta = 0$ is a branch point $\Rightarrow$
$z=\infty$ branch point.

(1d) $\sin\left(\frac{1}{1-z}\right)$: essential singularity at $z=1$.
no other finite singularities. $z=\infty$ is regular.

(1e) $1-e^z = -\left(z+\frac{1}{2}z^2+\cdots\right) = -z\left(1+\frac{1}{2}z+\cdots\right)$
$\Rightarrow (1-e^z)^{-3}$ has pole of order 3 at $z = 0 + 2n\pi i$
($z=0$ follow from above; $2n\pi i$ follow from periodicity of e^z).

(1f) $\frac{\sin\sqrt{z}}{\sqrt{z}}$ is entire: since $\sin z$ is an odd function, sign changes in square root cancel outs.
And $\lim_{z\to 0}\frac{\sin z}{z} = 1$, so $f(z)$ is analytic at 0
(More precisely: $\sin z = z - \frac{z^3}{3!} + \cdots = z\left(1-\frac{z^2}{3!}+\cdots\right)$, $|z|<\infty$
simple zero $\Rightarrow \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \cdots$ has a ~~Taylor~~ Laurent series valid for $0<|z|<\infty$. Since $\lim z\to 0$ exist, can extend to $|z|<\infty$. Thus
$$\frac{\sin\sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \cdots$$ is entire.)

(2) (a) $f(z) = \dfrac{z e^{iz}}{(z-\pi)^2}$; pole order 2 at $z=\pi$

$$\operatorname{Res} f(z) = \lim_{z\to\pi} \frac{1}{1!}\frac{d}{dz}\left[\cancel{(z-\pi)^2}\cdot \frac{z e^{iz}}{\cancel{(z-\pi)^2}}\right] = \left(e^{iz} + i z e^{iz}\right)_{z=\pi} = -(1+i\pi)$$

(b) $f(z) = \dfrac{1}{1-e^z}$ has poles, order 1, at $z = 2n\pi i$

$$\operatorname{Res} f(2n\pi i) = \lim_{z\to 2n\pi i} (z-2n\pi i)\cdot\frac{1}{1-e^z} = \left.\frac{1}{-e^z}\right|_{2n\pi i} = -1$$

↳ l' Hospital.

(c) $\dfrac{1}{\sin z}$ has pole, order 1, at $z = n\pi i$

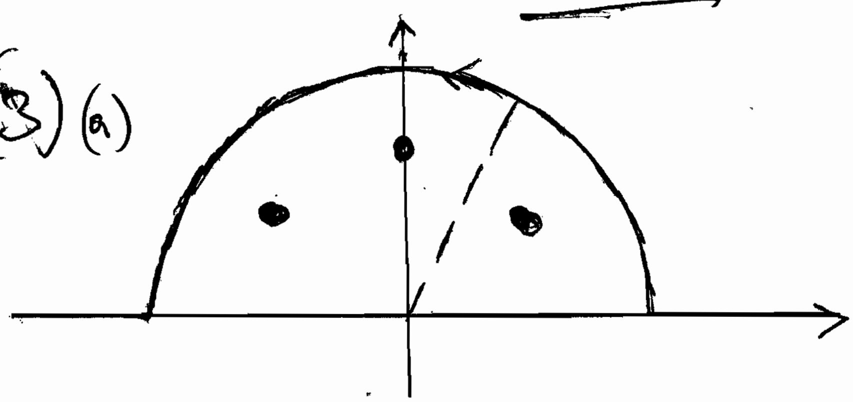
$$\operatorname{Res} f(n\pi i) = \lim_{z\to n\pi i} (z-n\pi i)\frac{1}{\sin z} = \left.\frac{1}{\cos z}\right|_{n\pi i} = (-1)^n$$

↳ L'Hospital

(d) $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots \frac{1}{n! z^n} + \cdots$

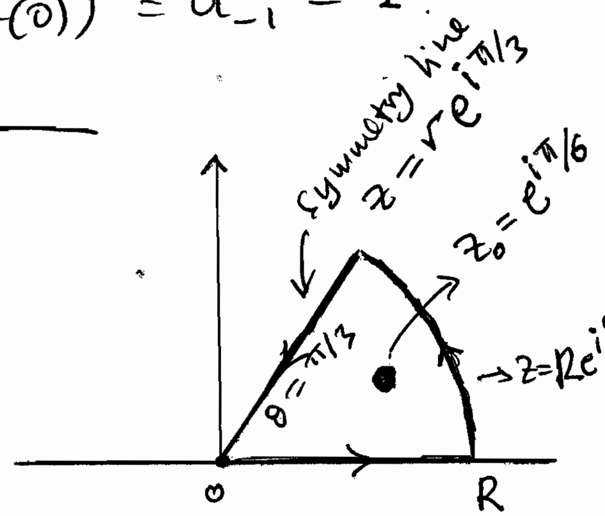
$z=0$: essential singularity, $\operatorname{Res}(f(0)) = a_{-1} = 1$.

(3) (a)



This contour leads to sum over 3 residues

(I)



This contour requires only one residue evaluation

(II)

We will work with contour II.

$$\oint_C \frac{z^4\,dz}{z^6+1} = \int_0^R \frac{x^4\,dx}{x^6+1} - \int_0^R \frac{r^4 e^{i\frac{4\pi}{3}}\cdot e^{i\frac{\pi}{3}}\,dr}{r^6+1}$$

$$+ \int_0^{\pi/3} \cancel{\frac{R^4 e^{i4\theta}\, iRe^{i\theta}\,d\theta}{R^6 e^{i6\theta}+1}}$$

0 as $R \to \infty$ (denom. is degree 6, numer is degree 4)

$$= 2\pi i \cdot \mathrm{Res}\Big|_{z=e^{i\pi/6}} = 2\pi i \cdot \frac{4z^4}{6z^5}\Big|_{z=e^{i\pi/6}} =$$

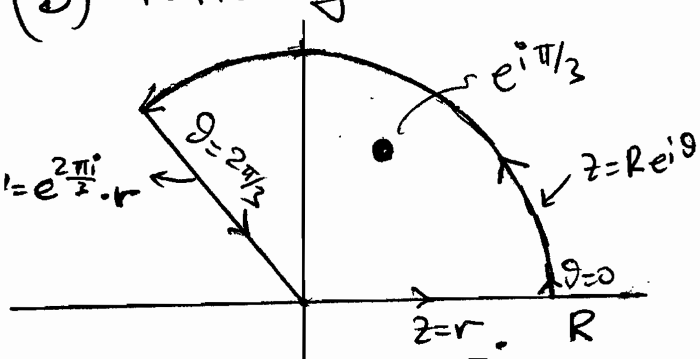
$$= 2\pi i \cdot \frac{2}{3} e^{-i\pi/6}$$

$$\Rightarrow \left(1 - e^{i5\pi/3}\right) \int_0^\infty \frac{x^4\,dx}{x^6+1} = 2\pi i \cdot \frac{2}{3} e^{-i\pi/6}$$

$$\Rightarrow I = \frac{2}{3} \cdot \frac{\pi}{\frac{e^{i\pi/6} - e^{11\pi i/6}}{2i}} = \frac{2}{3} \frac{\pi}{\underbrace{\sin \pi/6}_{=1/2}} = \frac{\pi}{3}$$

since $11\pi i/6 = 2\pi i - \frac{\pi i}{6}$

(b) Following discussion in class, (p. 9.3)



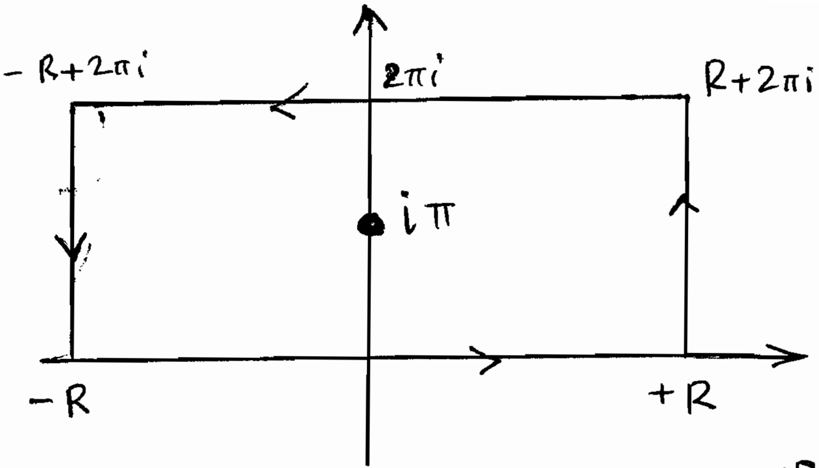
$$\left(1 - e^{i\frac{2\pi}{3}}\right) I = 2\pi i\, \mathrm{Res}\left(e^{i\pi/3}\right)$$

$$= 2\pi i \cdot \frac{1}{3e^{2i\pi/3}}$$

$$\Rightarrow I = \frac{1}{3} \cdot \frac{2\pi i}{e^{2i\pi/3} - e^{4\pi i/3}}$$

$$\Rightarrow I = \frac{\pi}{3} \cdot \frac{-2i e^{\pi i}}{e^{-i\pi/3} - e^{\pi i/3}} = \frac{\pi}{3} \frac{1}{\sin \pi/3} = \frac{2\pi}{3\sqrt{3}}$$

(c)



$(0<a<1)$

$$\oint \frac{e^{az}}{e^z+1}\,dz = 2\pi i \cdot \lim_{z\to i\pi}\left((z-i\pi)\cdot\frac{e^{az}}{e^z+1}\right) = \left(e^{az}\cdot\frac{1}{e^z}\right)\Big|_{i\pi}$$

$$= 2\pi i\, e^{ai\pi}/e^{i\pi} = -2\pi i\, e^{ia\pi}.$$

$$\|$$

$$\int_{-R}^{R}\frac{e^{ax}}{e^x+1}\,dx + \int_{R}^{-R}\frac{e^{a(x+2\pi i)}}{e^{x+2\pi i}+1}\,dx + \underbrace{\int_\uparrow + \int_\downarrow}_{\text{do these last.}}$$

The first two combine:

$$\int_{-\infty}^{\infty}\frac{e^{ax}\,dx}{e^x+1}\left(1-e^{2a\pi i}\right) = -e^{ia\pi} \Rightarrow I = \frac{-2\pi i}{e^{-ia\pi}-e^{ia\pi}} = \frac{\pi}{\sin a\pi}.$$

To show $\uparrow$, $\downarrow$ contribute nothing as $R\to 0$; Consider

$$\uparrow:\left|\int_{y=0}^{R}\frac{e^{a(R+iy)}}{e^{R+iy}+1}\,i\,dy\right| \le \int_{y=0}^{R}\frac{e^{aR}\,dy}{e^R-1} = \frac{Re^{aR}}{e^R-1}\to 0$$

as $R\to\infty$ since $a<1$.

($a>0$ ensured that $\sin a\pi \neq 0$).

(solutions to problems 4 & 5 will follow)