Solutions Set 7

(1) $y'' - \left(\frac{2x}{1-x^2}\right) y' + \frac{\alpha(\alpha+1)}{1-x^2} y = 0$ $\qquad$ +.1

Origin is a regular point, so try power series

$$y = \sum_0^\infty a_n x^n$$

$$y' = \sum_0^\infty a_n n x^{n-1} \qquad xy' = \sum_0^\infty a_n n x^n$$

$$y'' = \sum_0^\infty a_n n(n-1) x^{n-2} \qquad x^2 y'' = \sum_0^\infty a_n n(n-1) x^n$$

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$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1) y = 0 \Rightarrow$$

$$\sum_0^\infty n(n-1) a_n x^{n-2} + \sum_0^\infty \left\{ -a_n n(n-1) - 2a_n n + \alpha(\alpha+1) a_n \right\} x^n = 0$$

$$\Rightarrow \sum_0^\infty \left\{ (n+2)(n+1) a_{n+2} + [\alpha(\alpha+1) - n(n+1)] a_n \right\} x^n = 0$$

$$\Rightarrow a_{n+2} = -\frac{\alpha(\alpha+1) - n(n+1)}{(n+1)(n+2)} a_n$$

(i) $\alpha = 2m$; then let $a_0$ arbitrary; $a_1 = 0$

Then $a_{2k+1} = 0$, $k = 1, 2, \ldots$ (all odd terms vanish)

while $a_{2m+2} = \frac{[\overbrace{\alpha(\alpha+1) - 2m(2m+1)}^{=0}]}{(2m+1)(2m+2)} a_{2m} = 0$

and $a_{2k+2} = 0$, $k = m, m+1, \ldots$

i.e. only $(a_0, a_2, \ldots, a_{2m}) \neq 0$; solution is polynomial of degree $2$

4.2)

$\alpha = 0: \quad a_0 = 1, \quad a_2 = a_4 = \dots = 0$

$\alpha = 2: \quad a_0 = 1; \quad a_2 = -\frac{2\cdot 3 - 0}{1\cdot 2} a_1 = 3 \; ; \quad a_4 = a_6 = \dots = 0$

$\alpha = 4: \quad a_0 = 1; \quad a_2 = -\frac{4\cdot 5 - 0}{1\cdot 2} a_1 = 10$

$$a_4 = -\frac{4.5 - 2.3}{3.4} a_2 = +\frac{14}{12}\cdot 10 = \frac{35}{3}, \quad a_6 = \dots = 0$$

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So: $\tilde{P}_0(x) = 1, \quad \tilde{P}_2(x) = 1 - 3x^2$

$$\tilde{P}_4(x) = 1 - 10x^2 + \frac{35}{3}x^4$$

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If $\alpha = 2m+1$, then set $a_0 = 0$ ($\Rightarrow a_2 = a_4 = \dots = 0$)

and $a_1 = 1$; Find

$$\underbrace{a_{2m+3}}_{n+2} = -\frac{\overset{\alpha}{\overset{\shortparallel}{(2m+1)}}(2m+2) - \overset{n}{(2m+1)}\overset{n+1}{(2m+2)}}{\underset{n+1}{(2m+2)}\underset{n+2}{(2m+3)}} \, a_{2m+1} = 0$$

and $a_{2k+3} = 0, \quad k = m, m+1, \dots$

$\alpha = 1: \quad a_1 = 1, \quad a_3 = \dots = 0$

$\alpha = 3: \quad a_1 = 1, \quad \underset{n+2}{\underset{\shortparallel}{a_3}} = -\frac{3\cdot 4 - 1\cdot 2}{2\cdot 3} a_1 = -\frac{5}{3}, \quad a_5 = \dots = 0$

$\alpha = 5: \quad a_1 = 1, \quad a_3 = -\frac{5\cdot 6 - 1\cdot 2}{2\cdot 3}\cdot a_1 = -\frac{14}{3}$

$$a_5 = -\frac{5.6 - 3.4}{4.5} a_3 = \frac{18}{20}\cdot\frac{14}{3} = \frac{21}{5}$$

So $\tilde{P}_1(x) = x$, $P_3(x) = x - \frac{5}{3}x^3$

$\tilde{P}_5(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

To get $P_n(1) = 1$:

$P_0(x) = C_0 \tilde{P}_0(x) = \boxed{1 = P_0(x)}$

$P_1(x) = C_1 \tilde{P}_1(x) = C_1 x \Rightarrow C_1 \cdot 1 = 1 \Rightarrow C_1 = 1$

$\boxed{P_1(x) = x}$

$\tilde{P}_2(x) = C_2(1 - 3x^2) \Rightarrow C_2(1-3) = 1 \Rightarrow C_2 = -\frac{1}{2}$

$\boxed{P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}}$

$\tilde{P}_3(x) = C_3\left(x - \frac{5}{3}x^3\right) \Rightarrow C_3\left(1 - \frac{5}{3}\right) = 1 \Rightarrow C_3 = -\frac{3}{2}$

$\boxed{P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x}$

(2) let $z = x-1 \Rightarrow x = z+1$:

7.4

$$y'' - (z+1)y = 0 \quad : \text{ regular point}$$

$$y = \sum_0^\infty a_n \cancel{x}^n, \quad zy = \sum_0^\infty a_n z^{n+1}$$

$$y'' = \sum_0^\infty n(n-1)a_n z^{n-2}$$

$$\Rightarrow \sum_0^\infty (n+2)(n+1)a_{n+2} z^n - \sum_0^\infty a_{n-1} z^n - \sum_0^\infty a_n z^n = 0$$

$$\Rightarrow \sum_0^\infty \left\{(n+2)(n+1)a_{n+2} - a_n - a_{n-1}\right\} z^n = 0$$

With $a_{-1} = 0$: $\quad a_{n+2} = \dfrac{a_n + a_{n-1}}{(n+1)(n+2)}$ ; $n = 0, 1, \ldots$

So: leave $a_0, a_1$ arbitrary. Get two distinct solutions: $(a_0 = 1, a_1 = 0)$ : $y_1$

$(a_0 = 0, a_1 = 1)$ : $y_2$

$y_1$: $\quad a_2 = \dfrac{a_0 + a_{-1}}{2} = \dfrac{1}{2}$ ; $\quad a_3 = \dfrac{a_1 + a_0}{6} = \dfrac{1}{6}$

$$y_1(z) = 1 + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots \quad (z = x-1)$$

$y_2$: $\quad a_2 = \dfrac{a_0 + a_{-1}}{2} = 0, \quad a_3 = \dfrac{a_1 + a_0}{6} = \dfrac{1}{6}, \quad a_4 = \dfrac{a_2 + a_1}{12} \quad \dfrac{1}{12}$

$$y_2(z) = z + \frac{1}{6}z^3 + \frac{1}{12}z^4 + \ldots$$

7.5

(3) (a) $u'' + \frac{1}{z}u' - \frac{1}{z}u = 0$

$\quad\quad\quad\quad\frac{1}{z}(1+0z+\cdots) \quad\quad \frac{1}{z^2}(0 - z + 0z^2 + \cdots)$

$c(c-1) + p_0 c + q_0 = 0 \Rightarrow c(c-1)+c=0 \Rightarrow c^2=0$

$\Rightarrow c=0$ double

At $\infty$: $\frac{d^2w}{d\zeta^2} - \left(\frac{P}{\zeta^2} - \frac{2}{\zeta}\right)\frac{dw}{d\zeta} + \frac{Q}{\zeta^4}w = 0$

$\left.\begin{aligned}\frac{P}{\zeta^2} - \frac{2}{\zeta} &= \frac{1}{\zeta^2}\cdot\left(\frac{1}{1/\zeta}\right) - \frac{2}{\zeta} = -\frac{1}{\zeta} \\ \frac{Q}{\zeta^4} &= \frac{\left(-1/(1/\zeta)\right)}{\zeta^4} = -\frac{1}{\zeta^3}\end{aligned}\right\}$ irregular singularity

(b) $u'' - \left(\frac{1+z}{z}\right)u' + 2\frac{(1-z)}{z}u = 0 \quad\quad p_0=-1, q_0=0$

$z=0$: regular s.p.

$c(c-1) - c + 0 = 0 \Rightarrow c(c-2)=0$; $c=0,2$.

$z=\infty$: $P(\frac{1}{\zeta}) = p(\frac{1}{\zeta}) = -\frac{1}{(1/\zeta)} - 1 = -1-\zeta$

$\frac{P}{\zeta^2} - \frac{2}{\zeta} = -\frac{1+\zeta}{\zeta^2} - \frac{2}{\zeta}$ : irregular

(c) $$u'' - \frac{1}{z(2+z^2)}u' - \frac{6}{2+z^2}u = 0$$

7.6

$z = 0$, $\pm i\sqrt{2}$ : regular singularities.

$$P(\zeta) = p\left(\frac{1}{\zeta}\right) = \frac{1}{1/\zeta\,(2+1/\zeta^2)} = \frac{\zeta^3}{1+2\zeta^2}$$

$$\frac{P(\zeta)}{\zeta^2} - \frac{2}{\zeta} = \frac{\zeta}{1+2\zeta^2} - \frac{2}{\zeta} \longrightarrow O\left(\frac{1}{\zeta}\right)$$

$$Q(\zeta) = \frac{6}{2+1/\zeta^2} = \frac{6\zeta^2}{1+2\zeta^2}$$

$$\frac{Q(\zeta)}{\zeta^4} = \frac{6}{\zeta^2(1+2\zeta^2)} \longrightarrow O\left(\frac{1}{\zeta^2}\right).$$

$z = \infty$ is regular singular point.

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(5) (a) $$e^x = 1 + \cancel{x} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

~~Let Indicial equation~~ $x = 0$ ordinary point.

Try $$y_p = z^2(a_0 + a_1 z + \cdots) = \sum_{n=0}^{\infty} a_n z^{n+2}$$

$$y_p'' = \sum_{n=0}^{\infty} a_n (n+2)(n+1) z^n$$

So: $$\sum_{n=0}^{\infty} a_n (n+2)(n+1) z^n - \sum_{0}^{\infty} a_n z^{n+2} = \sum_{0}^{\infty} \frac{z^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{\infty}\left\{ a_n(n+2)(n+1) - a_{n-2} - \frac{1}{n!}\right\} z^n = 0$$ (7.7

$a_{-2} = a_{-1} = 0$

$$\Rightarrow a_n = \frac{a_{n-2} + 1/n!}{(n+1)(n+2)} = \frac{a_{n-2}}{(n+1)(n+2)} + \frac{1}{(n+2)!}$$

$$a_0 = \frac{1}{2!}, \quad a_1 = \frac{1}{3!}, \quad a_2 = \frac{a_0}{3\cdot 4} + \frac{1}{4!} = \frac{2}{4!}$$

$$y_p = z^2\left(\frac{1}{2!} + \frac{z}{3!} + \frac{2z^2}{4!} + \dots\right)$$

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(b) $x=0$ regular ; $y = \sum_0^{\infty} a_n z^{n+2}$

$zy = \sum_0^{\infty} a_n z^{n+3}$ ; $y'' = \sum_0^{\infty} a_n (n+2)(n+1) z^n$

$$\Rightarrow \sum_{n=0}^{\infty}\left[a_n(n+2)(n+1) + a_{n-3}\right] z^n = 1$$

$n=0$: $2a_0 = 1 \Rightarrow a_0 = 1/2$

$n=1$: $2.3\,a_1 = 0 \Rightarrow a_1 = 0$

$n=2$: $3.4\,a_2 = 0 \Rightarrow a_2 = 0$

$n=3$: $4.5\,a_3 + a_0 = 0 \Rightarrow a_3 = \frac{-a_0}{20} = \frac{-1}{40}$

$a_4 = a_5 = 0$

$n=6$: $7.8\,a_6 + a_3 = 0 \Rightarrow a_6 = \frac{-a_3}{56} = \frac{1}{2240}$

(7.8)

$$\therefore\ y_p(z) = {}^{1}z^2\left(\frac{1}{2} - \frac{z^3}{40} + \frac{z^6}{2240} + \right)\cdots$$

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(c) $z=0$ regular s.p. $\quad y'' - \frac{1}{x}y = 1 = z^0$

Indicial eqn: $\quad c(c-1) = 0 \ : \ c = 0, 1$

Since $\gamma = 0$, $\gamma + 2 = 2 \neq 0, 1$ :

let $\quad y = \cancel{z^2} \sum_0^\infty a_n z^{n+2}$

$$z y'' = \sum_0^\infty a_n (n+2)(n+1) z^{n+1}$$

$zy'' - y = z \Rightarrow$

$$\sum_0^\infty \{a_n (n+2)(n+1) \cancel{z^{n+1}} - a_{n-1}\} z^{n+1} = z$$

$n=0$: $\quad 2a_0 = 0 \Rightarrow a_0 = 0$

$n=1$ : $\quad 2.3\, a_1 - \overset{=0}{\cancel{a_0}} = 1 \Rightarrow a_1 = \frac{1}{6}$

$n=2$ : $\quad 3\cdot 4\, a_2 - a_1 = 0 \Rightarrow a_2 = \frac{a_1}{12} = \frac{1}{72}$

$n=3$ : $\quad 4.5\, a_3 - a_2 = 0 \Rightarrow a_3 = \frac{a_2}{20} = \frac{1}{1440}$

$$y_p = z^2\left(\frac{z}{6} + \frac{z^2}{72} + \frac{z^3}{1440}\right)\cdots$$

(d) $y'' + \frac{1}{x^2} y = x^{-5/2} e^x = x^{-5/2} \sum_0^\infty \frac{x^n}{n!}$

7.9

$x=0$ regular; $\gamma = -5/2$

Indicial eqn: $p_0 = 0$, $q_0 = 1$

$$c(c-1) + q_0 = 0 \Rightarrow c^2 - c + q_0 = 0$$

$$\Rightarrow c = \frac{1 \pm i\sqrt{3}}{2}; \quad \gamma + n = n - \frac{5}{2} \text{ never}$$

a root of indicial eq.

So $y_p = z^{\gamma+2} \sum_0^\infty a_k z^k = \sum_0^\infty a_k z^{k-1/2}$

$$z^2 y'' = \sum_0^\infty a_k (k - \tfrac{1}{2})(k - \tfrac{3}{2}) z^{k-\frac{1}{2}}$$

$$z^2 y'' + y = \sum_0^\infty \left[a_k (k-\tfrac{1}{2})(k-\tfrac{3}{2}) + a_k\right] z^{k-\frac{1}{2}}$$

$$= z^{-1/2} e^z = \sum_{k=0}^\infty \frac{z^{k-1/2}}{k!}$$

$$\Rightarrow a_k = \frac{1}{k!} \cdot \frac{1}{(k-\frac{1}{2})(k-\frac{3}{2})+1} = \frac{1}{k!(k^2 - 2k + \frac{7}{4})}$$

$$a_0 = \frac{4}{7}, \quad a_1 = \frac{4}{3}, \quad a_2 = \frac{2}{7}$$

$$y_p = \frac{1}{\sqrt{z}} \left( \frac{4}{7} + \frac{4}{3} z + \frac{2}{7} z^2 + \dots \right)$$

(4 a) $zu'' + u' - u = 0$

7.10 $u'' + \frac{1}{z}u' - \frac{1}{z}u = 0$ ; $p_0 = 1$, $q_0 = 0$

Solution $u(z) = z^c \sum_0^\infty a_n z^n$

Find (indicial eqn.) $c(c-1) + c = 0 \Rightarrow c^2 = 0$

double root

So $u_1(z) = \sum_0^\infty a_n z^n$

$u_1' = \sum_{n=0}^\infty a_n n z^{n-1}$

$zu_1'' = \sum_{n=0}^\infty a_n n(n-1) z^{n-1}$

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$a_{-1} = 0$

$zu'' + u' - u = 0$

$\sum_{n=0}^\infty \left\{ a_n n(n-1) + a_n n - a_{n-1} \right\} z^{n-1} = 0$

$\Rightarrow a_{n+1} = \frac{a_n}{(n+1)^2} = \cdots = \frac{a_0}{[(n+1)!]^2}$

$u_1(z) = a_0 \sum_{n=0}^\infty \frac{z^k}{(k!)^2}$ ; let $a_0 = 1$.

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Second solution has the form

$u_2 = u_1 \log z + z\chi(z)$

$\chi(z) = \sum_0^\infty b_n z^n$

~~use log z =~~

<u>Substitute:</u>

$$u_2 = u_1 \log z + z\chi$$

$$u_2' = u_1' \log z + \frac{1}{z} u_1 + \chi + z\chi'$$

$$u_2'' = u_1'' \log z + 2u_1' \cdot \frac{1}{z} - \frac{u_1}{z^2} + 2\chi' + z\chi''$$

$$z u_2'' = z u_1'' \log z + 2u_1' - \frac{u_1}{z} + 2z\chi' + z^2\chi''$$

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$$z u_2'' + u_2' - u_2 = 0 \quad \Rightarrow$$

$$\log z \, \underbrace{\left(-u_1 + u_1' + z u_1''\right)}_{=0} +$$

$$2u_1' - \cancel{\frac{u_1}{z}} + 2z\chi' + z^2\chi''$$

$$+ \cancel{\frac{u_1}{z}} + \chi + z\chi'$$

$$- z\chi \qquad = 0$$

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$$z^2\chi'' + 3z\chi' + (1-z)\chi = -2u_1'$$

$$= -2 \sum_{k=0}^{\infty} \frac{k z^{k-1}}{(k!)^2}$$

$$= -2 \sum_{k=0}^{\infty} \frac{z^k}{k!(k+1)!}$$

7.12 The equation for $\chi(z)$ has indicial eqn

$$\chi'' + \frac{1}{z}\chi' + \frac{1-z}{z^2}\chi = -2z^{-2}\sum_0^\infty \frac{z^k}{k!(k+1)!}$$

$$\gamma = -2$$

$$c(c-1) + c + 1 = 0$$

$$c^2 + 1 = 0 \Rightarrow \underline{\underline{c = \pm i}} \qquad \text{so } \gamma + n \neq c.$$

Try $\chi = z^{\gamma+2}\sum b_k z^k = \sum_{k=0}^\infty b_k z^k$

$$(1-z)\chi = \sum_{k=0}^\infty b_k z^k - \sum_{k=0}^\infty b_k z^{k+1} \qquad \underline{b_{-2} = b_{-1} = 0}$$

$$= \sum_{k=0}^\infty (b_k - b_{k-1}) z^k$$

$$z\chi' = \sum_{k=0}^\infty b_k k z^k$$

$$z^2\chi'' = \sum_{k=0}^\infty b_k k(k-1) z^k$$

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$$z^2\chi'' + z\chi' + (1-z)\chi = -2\sum_{k=0}^\infty \frac{z^k}{k!(k+1)!}$$

$$\Rightarrow \sum_{k=0}^{\infty}{}' \left\{ b_k k(k-1) + b_k k + (b_k - b_{k-1}) + \frac{2}{k!(k+1)!} \right\} z^k = 0$$

$$\Rightarrow \quad b_k (k^2+1) - b_{k-1} + \frac{2}{k!\,(k+1)!} = 0$$ (7.13

$$\Rightarrow \boxed{b_k = \frac{b_{k-1}}{k^2+1} + \frac{2}{k!\,(k+1)!\,(k^2+1)}}$$

$$b_0 = 2 \;;\; b_1 = \frac{b_0}{2} + \frac{2}{4} = \frac{3}{2}, \text{ etc.}$$

— . —

(b) $z u'' - (1+z) u' + 2(1-z) u = 0$

$$\Rightarrow u'' - \frac{1+z}{z} u' + \frac{2(z-z^2)}{z^2} u = 0$$

$$q_0 = -1, \quad p_0 = 0$$

$$c(c-1) - c = 0 \Rightarrow c(c-2) = 0 \Rightarrow$$

$$c = 0, 2.$$

Try $c = 2$:

$$u_1 = z^2 \sum_0^\infty a_n z^n = \sum_0^\infty a_n z^{n+2}$$

$$2(1-z) u = \sum_0^\infty (2a_n z^{n+2} - 2a_n z^{n+3})$$

$$-(1+z) u' = -(1+z) \sum_0^\infty a_n (n+2) z^{n+1} =$$

$$= \sum - a_n (n+2) z^{n+1} - a_n (n+2) z^{n+2}$$

7.4)

$$zu'' = \sum_{0}^{\infty} a_n (n+2)(n+1) z^{n+1}$$

So

$$zu'' - (1+z)u' + 2(1-z)u = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} \{ a_n(n+2)(n+1) - a_n(n+2) - a_{n-1}(n+1) + 2a_{n-1} - 2a_{n-2} \} z^{n+1} = 0$$

$$(a_{-2} = a_{-1} = 0)$$

$$\Rightarrow a_n n(n+2) - a_{n-1}(n-1) - 2a_{n-2} = 0$$

$$a_n = \frac{a_{n-1}(n-1) + 2a_{n-2}}{n(n+2)}, \quad n = 1, 2, \ldots$$

$$a_1 = \frac{a_0 \cdot 0 + 0}{1 \cdot 3} = 0$$

$$a_2 = \frac{a_1 + 2a_0}{2 \cdot 4} = \frac{1}{4} a_0$$

$$a_3 = \frac{2a_2 + 2a_1}{3 \cdot 5} = \frac{1}{30} a_0$$

etc.

$$u_1(z) = a_0 z^2 \left(1 + \frac{z}{4} + \frac{z^2}{30} + \cdots\right)$$

Again, look for second solution in form:

$$u_2(z) = u_1 \log z + z^3 \chi(z)$$

Substituting:

$$u_2' = u_1' \log z + \frac{1}{z} u_1 + 3z^2 \chi + z^3 \chi'$$

$$u_2'' = u_1'' \log z + \frac{2}{z} u_1' - \frac{1}{z^2} u_1 + 6z\chi + 6z^2\chi' + z^3\chi''$$

$$z u_2'' = z u_1'' \log z + 2u_1' - \frac{1}{z} u_1 + 6z^2\chi + 6z^3\chi' + z^4\chi''$$

$$(1+z) u_2' = -(1+z) u_1' \log z - \frac{1+z}{z} u_1 - 3z^2(1+z)\chi - z^3(1+z)\chi'$$

$$2(1-z) u_2 = 2(1-z) u_1 \log z + 2(1-z) z^3 \chi(z)$$

$$= 0$$

$$\Rightarrow z^4 \chi'' - (1+z) z^3 \chi' + 6z^3\chi' + \left(6z^2 - 3z^2(1+z) + 2(1-z)z^3\right)\chi$$

$$+ 2u_1' - \frac{1}{z} u_1 - \frac{1+z}{z} u_1 = 0$$

7.16

$$\Rightarrow z^4 x'' + z^3(5-z)x' + 3z^2(3z-1)x$$

$$= 2u_1' - \frac{2}{z}u_1 - u_1$$

$$\Rightarrow \cancel{z}\, x'' + \frac{5-z}{z}x' + \frac{3(3z-1)}{z^2}x$$

$$= \frac{1}{z^4}\left(2u_1' - \frac{2+z}{z}u_1\right)$$

Since $u_1 = z^2 \sum a_k z^k \Rightarrow$

$$\text{rhs} = \frac{1}{z^4} \cdot z \sum_0^\infty b_k z^k = z^{-3} \sum_0^\infty b_k z^k$$

$$\gamma = -3$$

Indicial eqn for de. satisfied by $x$:

$$p_0 = 5, \; q_0 = -3$$

$$c(c-1) + 5c - 3 = 0$$

$$c^2 + 4c - 3 = 0$$

$$c = -2 \pm \sqrt{7} \neq \gamma + n$$

So $x(z) = z^{-1} \sum_{k=0}^{\infty} c_k z^k$

will work.