Mathematical Methods in Science and Engineering
Part II
MATH 466, Fall 2006

Jens Lorenz

December 5, 2006

Department of Mathematics and Statistics,
UNM, Albuquerque, NM 87131

Contents

5 Partial Differential Equations **3**
- 5.1 A Dirichlet Problem for Laplace's Equation . . . 3
- 5.2 The One–Way Wave Equation . . . 4
- 5.3 The Wave Equation in 1D . . . 4
- 5.4 The Wave Equation in 2D, Separation of Variables . . . 4
- 5.5 The Laplacian in Polar Coordinates . . . 5
- 5.6 Separation of Helmholtz' Equation in Polar Coordinates: Derivation of Bessel's Equation . . . 6
- 5.7 The Laplacian in 3D Spherical Coordinates . . . 7
- 5.8 Separation of Helmholtz' Equation in Spherical Coordinates: Derivation of the Spherical Bessel Equation and the Associated Legendre Equation . . . 7
  - 5.8.1 Derivation of the Equations . . . 7
  - 5.8.2 The Spherical Bessel Equation . . . 8
  - 5.8.3 Legendre's Equation . . . 9
- 5.9 Legendre Polynomials . . . 10
- 5.10 Solution of the Associated Legendre Equation . . . 14
- 5.11 Spherical Harmonics as Eigenfunctions . . . 20
- 5.12 Spherical Harmonics are Restrictions of Harmonic Polynomials to the Unit Sphere . . . 23

6 Second Order Linear Homogeneous ODEs **26**
- 6.1 Ordinary Points, Regular Singular Points, Irregular Singular Points 26
- 6.2 Equations with Constant Coefficients . . . 26
- 6.3 Cauchy–Euler Equations . . . 28
- 6.4 Series Solutions Near an Ordinary Point . . . 29
- 6.5 Solutions Near a Regular Singular Point . . . 31
- 6.6 Bessel Functions of the First Kind . . . 33

6.7 Bessel Functions of the Second Kind 34
6.8 Recurrence Relations 36
6.9 Liouville Transformation 37
6.10 Sturm's Comparison Theorem 38

5 Partial Differential Equations

5.1 A Dirichlet Problem for Laplace's Equation

We know that the heat equation

$$u_t = \kappa \Delta u, \quad u(x,y,0) = u_0(x,y)$$

models the time evolution of temperature. In the following, we consider the stationary equation, $\Delta u = 0$, where u is prescribed on the boundary of the disk. This problems leads in a natural way to Fourier expansion of the boundary function.

Let

$$B_1 = \{(x,y) : x^2 + y^2 < 1\}$$

denote the unit circle.

The problem

$$\Delta u = 0 \quad \text{in} \quad B_1, \quad u = f \quad \text{on} \quad \partial B_1$$

leads to Fourier expansion of $f(\phi)$.

We have

$$\Delta u = \frac{1}{r}(r u_r)_r + \frac{1}{r^2}\, u_{\phi\phi} \ .$$

The ansatz

$$u(r,\phi) = R(r)\Phi(\phi)$$

yields

$$\Phi'' + m^2 \Phi = 0, \quad r^2 R'' + rR' - m^2 R = 0 \ .$$

Obtain

$$\Phi(\phi) = c_1 \cos m\phi + c_2 \sin m\phi \ .$$

The equation for R is an Euler equation. The ansatz $R = r^\lambda$ leads to

$$\lambda_1 = m, \quad \lambda_2 = -m \ .$$

The terms r^{-m} are singular for $m \geq 1$, and will be used for an exterior Dirichlet problem. If

$$f(\phi) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty} A_m \cos m\phi + \sum_{m=1}^{\infty} B_m \sin m\phi \ ,$$

then

$$u(r,\phi) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty} A_m r^m \cos m\phi + \sum_{m=1}^{\infty} B_m r^m \sin m\phi \ ,$$

We have

$$
\begin{aligned}
A_j &= \frac{1}{\pi}\int_0^{2\pi} \cos j\phi\, f(\phi)\, d\phi, \quad j = 0, 1, \ldots \\
B_j &= \frac{1}{\pi}\int_0^{2\pi} \sin j\phi\, f(\phi)\, d\phi, \quad j = 1, 2, \ldots
\end{aligned}
$$

5.2 The One–Way Wave Equation

The initial value problem

$$u_t + a u_x = 0, \quad u(x,0) = f(x)$$

is solved by

$$u(x,t) = f(x - at) \ .$$

This describes the propagation of $f(x)$ with speed a.

5.3 The Wave Equation in 1D

The ivp

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = g(x), \quad u_t(x,0) = h(x) \ ,$$

is solved by

$$u(x,t) = \frac{1}{2}\Big(g(x+ct) + g(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy \ .$$

This is called d'Alembert's formula.

5.4 The Wave Equation in 2D, Separation of Variables

Consider the equation

$$u_{tt} = c^2 \Delta u, \quad \Delta u = u_{xx} + u_{yy} \ .$$

The ansatz

$$u(x,y,t) = \alpha(t)\psi(x,y)$$

leads to

$$\frac{\alpha''(t)}{c^2\alpha(t)} = \frac{\Delta\psi(x,y)}{\psi(x,y)} =: -k^2 \ .$$

Obtain

$$\alpha(t) = c_1 e^{ickt} + c_2 e^{-ickt} \ .$$

For ψ obtain Helmholtz' equation

$$\Delta\psi(x,y) + k^2\psi(x,y) = 0 \ .$$

Note: The choice of the constant as $-k^2$ leads to oscillatory functions in time and space; exponential growth in time and space is physically unreasonable.

Let

$$\psi(x,y) = X(x)Y(y) \ .$$

Obtain

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + k^2 = 0 \ .$$

If $k_1^2 + k_2^2 = k^2$ and

$$\begin{aligned} X'' + k_1^2 X &= 0 \\ Y'' + k_2^2 Y &= 0 \end{aligned}$$

then $\psi(x,y) = X(x)Y(y)$ solves Helmholtz' equation.

Example: Give an initial condition

$$u(x,y,0) = \cos(x+2y), \quad u_t(x+2y) = 0 \ .$$

Solution:

$$u(x,y,t) = \cos(x+2y)\cos(c\sqrt{5}t) \ .$$

Note that one needs the wave vectors

$$\mathbf{k} = (k_1,k_2) = (1,2)$$

and $-\mathbf{k}$.

5.5 The Laplacian in Polar Coordinates

If $\psi(x,y)$ is a given function in Cartesian coordinates (x,y), then the corresponding function in polar coordinates (ρ,ϕ) is

$$\tilde{\psi}(\rho,\phi) = \psi(\rho\cos\phi, \rho\sin\phi) \ .$$

If $f = \Delta\psi$ then $\tilde{f}$ can be obtained from $\tilde{\psi}$ as follows:

$$\begin{aligned} \tilde{f} &= \tilde{\psi}_{\rho\rho} + \frac{1}{\rho}\tilde{\psi}_\rho + \frac{1}{\rho^2}\tilde{\psi}_{\phi\phi} \\ &= \frac{1}{\rho}\Big(\rho\tilde{\psi}_\rho\Big)_\rho + \frac{1}{\rho^2}\tilde{\psi}_{\phi\phi} \end{aligned}$$

Obtain

$$\Delta_{polar} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} \ .$$

5.6 Separation of Helmholtz' Equation in Polar Coordinates: Derivation of Bessel's Equation

The equation for $\psi(\rho,\phi)$ is

$$\frac{1}{\rho}(\rho\psi_\rho)_\rho + \frac{1}{\rho^2}\,\psi_{\phi\phi} + k^2\psi = 0\ .$$

Ansatz:

$$\psi(\rho,\phi) = R(\rho)\Phi(\phi)$$

To be physically meaningful, $R(\rho)$ must be defined for $\rho > 0$ and $\Phi(\phi)$ must have period 2π. Obtain

$$\frac{1}{\rho R}(\rho R')' + \frac{\Phi''}{\rho^2\Phi} + k^2 = 0\ .$$

Multiply by ρ^2,

$$\frac{\rho}{R}(\rho R')' + \frac{\Phi''}{\Phi} + \rho^2 k^2 = 0\ .$$

Obtain

$$\Phi'' + m^2\Phi = 0$$

and

$$\frac{\rho}{R}(\rho R')' + \rho^2 k^2 - m^2 = 0\ . \tag{5.1}$$

The general solution of the Φ equation is

$$\Phi(\phi) = c_1\cos(m\phi) + c_2\sin(m\phi)\ ,$$

and 2π – periodicity of Φ requires m to be integer. In the R equation let

$$x = k\rho, \quad y(x) = R(x/k)$$

where $k > 0$ is fixed. For $y(x)$ obtain Bessel's equation of index m,

$$x^2y''(x) + xy'(x) + (x^2 - m^2)y(x) = 0\ .$$

We seek solutions defined for $x > 0$.

If $y(x)$ solves Bessel's equation of index m, then

$$\Psi(\rho,\phi) = y(k\rho)\Big(c_1\cos(m\phi) + c_2\sin(m\phi)\Big)$$

solves Helmholtz' equation with eigenvalue k^2 and

$$u(\rho,\phi,t) = \Big(\beta_1\cos(ckt) + \beta_2\sin(ckt)\Big)\Psi(\rho,\phi)$$

solves the wave equation.

5.7 The Laplacian in 3D Spherical Coordinates

Denote spherical coordinates by r, θ, ϕ where

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

The angle θ is called the polar angle whereas ϕ is the azimuthal angle.

The Laplacian applied to $\psi(r, \theta, \phi)$ is

$$\Delta\psi = \frac{1}{r^2 \sin\theta} \Big(\sin\theta (r^2 \psi_r)_r + (\sin\theta\, \psi_\theta)_\theta + \frac{1}{\sin\theta}\, \psi_{\phi\phi} \Big) \ .$$

5.8 Separation of Helmholtz' Equation in Spherical Coordinates: Derivation of the Spherical Bessel Equation and the Associated Legendre Equation

Consider the 3D wave equation, $u_{tt} = c^2 \Delta u$. The ansatz

$$u(r, \theta, \phi, t) = e^{i\omega t} \psi(r, \theta, \phi)$$

leads to

$$\Delta\psi + \Big(\frac{\omega}{c}\Big)^2 \psi = 0 \ .$$

In other words, we obtain Helmholtz' equation

$$\Delta\psi + k^2 \psi = 0, \quad k = \pm\frac{\omega}{c} \ ,$$

and every solution ψ, k gives the solutions

$$u = \Big(A_k e^{-ikct} + B_k e^{ikct} \Big) \psi(r, \theta, \phi)$$

of the wave equation. Here k (with $[k] = 1/length$) is the wave number and $\omega = kc$ (with $[\omega] = 1/time$) is the frequency of the solution u.

We want to discuss solutions $\psi(r, \theta, \phi)$ of Helmholtz' equation that are obtained by separation of variables in spherical coordinates. We will see: The r–dependence leads to a modification of Bessel's equation, the so–called spherical Bessel equation. As in 2D, the ϕ–dependence leads to the oscillator equation $\Phi''(\phi) + m^2 \Phi(\phi) = 0$. In θ–direction one obtains Legendre's equation for $m = 0$ and the so–called associated Legendre equation for $m = \pm 1, \pm 2, \ldots$.

5.8.1 Derivation of the Equations

Let

$$\psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \ .$$

Substitute this ansatz into

$$\Delta\psi + k^2\psi = 0$$

and divide by ψ to obtain

$$\frac{1}{Rr^2}(r^2R_r)_r + \frac{1}{\Theta r^2 \sin\theta}(\sin\theta\,\Theta_\theta)_\theta + \frac{1}{\Phi r^2 \sin^2\theta}\Phi_{\phi\phi} + k^2 = 0\ . \tag{5.2}$$

Multiply by $r^2\sin^2\theta$.

Obtain that

$$\frac{\Phi''}{\Phi} = const =: -m^2\ .$$

Here m must be an integer to make $\Phi(\phi)$ periodic with period 2π.

Substituting $\Phi''/\Phi = -m^2$ into (5.2) one obtains

$$\frac{1}{R}(r^2R_r)_r + \frac{1}{\Theta\sin\theta}(\sin\theta\,\Theta_\theta)_\theta - \frac{m^2}{\sin^2\theta} + r^2k^2 = 0\ . \tag{5.3}$$

There are two terms depending only on r and two terms depending only on θ. Call the separation constant Q. Obtain

$$\frac{1}{\sin\theta}(\sin\theta\,\Theta')' + \Big(Q - \frac{m^2}{\sin^2\theta}\Big)\Theta = 0\ . \tag{5.4}$$

and

$$(r^2R')' + (r^2k^2 - Q)R = 0\ . \tag{5.5}$$

The R–equation

$$r^2R'' + 2rR' + (k^2r^2 - Q)R = 0 \tag{5.6}$$

is called a spherical Bessel equation. (The only difference to the R–equation (5.1) that one obtains in 2D is the factor 2 in the equation above.)

5.8.2 The Spherical Bessel Equation

First consider (5.6) for $k = 0$. One obtains an Euler equation and the ansatz

$$R(r) = r^\lambda$$

leads to the indicial equation

$$\lambda(\lambda+1) = Q\ .$$

The θ–equation will require to choose

$$Q = Q_n = n(n+1), \quad n = 0, 1, \ldots$$

For $Q = n(n+1)$ the indicial equation has the roots

$$\lambda_1 = n, \quad \lambda_2 = -n-1\ .$$

This yields the general solution

$$R(r) = \alpha r^n + \frac{\beta}{r^{n+1}}$$

of (5.6) for $k = 0$ and $Q = n(n+1)$.

Now consider (5.6) for $k > 0$. One can transform to Bessel's equation as follows: Define

$$x = kr, \quad y(x) = y(kr) = r^{1/2} R(r) \ .$$

(Note that the factor $r^{1/2}$ was not present in 2D.) Obtain:

$$\begin{aligned} R(r) &= r^{-1/2} y(kr) \\ R'(r) &= -\frac{1}{2} r^{-3/2} y(kr) + k r^{-1/2} y'(kr) \\ R''(r) &= \frac{3}{4} r^{-5/2} y(kr) - k r^{-3/2} y'(kr) + k^2 r^{-1/2} y''(kr) \end{aligned}$$

Therefore, if $R(r)$ satisfies (5.6), then we have

$$\begin{aligned} 0 &= r^{1/2}\Big(r^2 R'' + 2rR' + (k^2r^2 - Q)R\Big) \\ &= k^2 r^2 y''(kr) - rky'(kr) + \frac{3}{4} y(kr) + 2kry'(kr) - y(kr) + (k^2r^2 - Q)y(kr) \\ &= x^2 y''(x) + xy'(x) + (x^2 - Q - \frac{1}{4})y(x) \end{aligned}$$

We have derived the equation

$$x^2 y''(x) + xy(x) + \Big(x^2 - Q - \frac{1}{4}\Big)y(x) = 0 \ ,$$

which is Bessel's equation.

If $Q = n(n+1)$ then

$$Q + \frac{1}{4} = (n + \frac{1}{2})^2 \ ,$$

i.e., we obtain Bessel's equation of index $n + \frac{1}{2}$.

5.8.3 Legendre's Equation

The Θ equation (5.4) reads

$$\Theta'' + \frac{\cos\theta}{\sin\theta}\,\Theta' + \Big(Q - \frac{m^2}{\sin^2\theta}\Big)\Theta = 0 \ . \tag{5.7}$$

Recall that $0 < \theta < \pi$. Thus we may write

$$\begin{aligned} \Theta(\theta) &= P(\cos\theta) \\ \Theta'(\theta) &= -\sin\theta\, P'(\cos\theta) \\ \Theta''(\theta) &= -\cos\theta\, P'(\cos\theta) + \sin^2\theta\, P''(\cos\theta) \end{aligned}$$

If $\Theta(\theta)$ solves (5.7) and if $P(\cos\theta) = \Theta(\theta)$ then obtain

$$\sin^2\theta\, P''(\cos\theta) - 2\cos\theta\, P'(\cos\theta) + \Big(Q - \frac{m^2}{\sin^2\theta}\Big)P(\cos\theta) = 0 \ . \qquad (5.8)$$

Set $x = \cos\theta$. Obtain

$$(1-x^2)\, P''(x) - 2x\, P'(x) + \Big(Q - \frac{m^2}{1-x^2}\Big)P(x) = 0 \ . \qquad (5.9)$$

This equation is called an associated Legendre equation. The points $x = \pm 1$ are regular singular points. One can show that (5.9) has nontrivial solutions that are bounded for $-1 < x < 1$ if only if $Q = n(n+1)$ and $-n \le m \le n$ with integers m, n.

We now assume

$$Q = n(n+1)$$

with integer n, $n \ge 0$. Then, for $m = 0$, one obtains Legendre's equation

$$(1-x^2)\, P''(x) - 2x\, P'(x) + n(n+1)P(x) = 0 \ . \qquad (5.10)$$

Remark: If $m = 0$ then $\Phi(\phi) = const$, i.e., we consider solutions $\psi(r,\theta,\phi)$ of Helmholtz's equation that are independent of ϕ.

5.9 Legendre Polynomials

Lemma 5.1 *The n–th degree polynomial*

$$P(x) = D^n\Big((x^2-1)^n\Big), \quad D = \frac{d}{dx} \ ,$$

solves Legendre's equation (5.10).

Proof: Let

$$v = (x^2-1)^n, \quad v' = 2nx(x^2-1)^{n-1} \ ,$$

thus

$$(1-x^2)v' + 2nxv = 0 \ . \qquad (5.11)$$

Recall Leibniz' rule,

$$\begin{aligned} D^{n+1}(fg) &= \sum_{j=0}^{n+1} \begin{pmatrix} n+1 \\ j \end{pmatrix} (D^j f)(D^{n+1-j} g) \\ &= f D^{n+1} g + (n+1)(Df)(D^n g) + \frac{1}{2} n(n+1)(D^2 f)(D^{n-1} g) + \ldots \end{aligned}$$

Apply D^{n+1} to (5.11),

$$(1-x^2)D^{n+2}v-2x(n+1)D^{n+1}v-n(n+1)D^nv+2nxD^{n+1}v+2n(n+1)D^nv=0$$

thus

$$(1-x^2)D^{n+2}v-2xD^{n+1}v+n(n+1)D^nv=0$$

This shows that D^nv solves Legendre's equation and completes the proof. $\diamond$

The polynomial

$$P_n(x)=\frac{1}{n!2^n}\,D^n\Big((x^2-1)^n\Big) \tag{5.12}$$

is called the n–th Legendre polynomial. Formula (5.12) is called Rodrigues' formula for the Legendre polynomial $P_n(x)$ of degree n.

We claim that the normalization factor $1/(n!2^n)$ is chosen so that $P_n(1)=1$. In other words, we have

Lemma 5.2 *The n–th Legendre polynomial, defined by (5.12), satisfies*

$$P_n(1)=1\ .$$

Proof: We have

$$D^n\Big((x+1)^n(x-1)^n\Big)=\sum_{j=0}^n\binom{n}{j}D^j\Big((x+1)^n\Big)D^{n-j}\Big((x-1)^n\Big)$$

Evaluate at $x=1$. Note that, for $j\geq 1$, the term $D^{n-j}((x-1)^n)$ is zero at $x=1$. For $j=0$ obtain:

$$\begin{aligned}\Big(D^j((x+1)^n)\,D^{n-j}((x-1)^n)\Big)\Big|_{x=1} &= \Big((x+1)^nD^n((x-1)^n)\Big)\Big|_{x=1}\\ &= 2^n\,n!\ .\end{aligned}$$

This is the value of the above sum at $x=1$. The lemma is proved. $\diamond$

Using Rolle's theorem, it is easy to show:

Lemma 5.3 *The n–th Legendre polynomial $P_n(x)$ has n simple zeros in the open interval $-1<x<1$.*

For series expansions in terms of the $P_n(x)$ one needs to know orthogonality and the normalization constants.

Lemma 5.4 *The sequence of Legendre polynomials,*

$$P_n(x)=\frac{1}{n!2^n}D^n((x^2-1)^n),\quad n=0,1,\dots$$

satisfies

$$\int_{-1}^1 P_m(x)P_n(x)=\frac{2\delta_{mn}}{2n+1},\quad m,n=0,1,\dots$$

Proof: Orthogonality: For $m < n$ it follows through integration by parts that

$$\int_{-1}^{1} D^m\Big((x^2-1)^m\Big) D^n\Big((x^2-1)^n\Big)\, dx = 0 \; .$$

(Move D^n to the first factor through integration by parts.)

Normalization: We claim that

$$\int_{-1}^{1} \Big(P_n(x)\Big)^2\, dx = \frac{2}{2n+1} \; . \tag{5.13}$$

For the left–hand side in (5.13) we have

$$\begin{aligned} lhs &= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} D^n\Big((x^2-1)^n\Big) D^n\Big((x^2-1)^n\Big)\, dx \\ &= \frac{(2n)!}{2^{2n}(n!)^2}\, J \end{aligned}$$

with

$$J = \int_{-1}^{1} (1-x^2)^n\, dx \; .$$

To obtain last equation we have used n fold integration by parts, noting that

$$D^{2n}\Big((x^2-1)^n\Big) = (2n)! \; .$$

It remains to compute J. We will prove:

$$\int_{-1}^{1} (1-x^2)^n\, dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \; . \tag{5.14}$$

To show this, we will use Euler's *Beta* function and its relation to the Γ function.

By definition,

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt, \quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt,$$

for $p > 0, q > 0, z > 0$.

Lemma 5.5 *For all $p > 0, q > 0$,*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \; .$$

A proof is given below.

Using the substitution

$$x^2 = y, \quad 2x\, dx = dy, \quad dx = \frac{1}{2} y^{-1/2}\, dy$$

we have

$$\begin{aligned}
J &= 2\int_0^1 (1-x^2)^n \, dx \\
&= \int_0^1 y^{-1/2}(1-y)^n \, dy \\
&= B(\frac{1}{2}, n+1) \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}
\end{aligned}$$

Here $\Gamma(n+1) = n!$. Also, using the fundamental functional equation for the Γ function, $\Gamma(z+1) = z\Gamma(z)$,

$$\begin{aligned}
\Gamma(\frac{1}{2}+1) &= \frac{1}{2}\Gamma(\frac{1}{2}) \\
\Gamma(\frac{1}{2}+2) &= \Gamma(\frac{3}{2}+1) = \frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}) \\
\Gamma(\frac{1}{2}+n+1) &= \frac{1\cdot 3\cdot \ldots \cdot (2n+1)}{2^{n+1}}\Gamma(\frac{1}{2})
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} &= \frac{2^{n+1}}{1\cdot 3\cdot \ldots \cdot (2n+1)} \\
&= \frac{2^{2n+1}n!}{(2n+1)!}
\end{aligned}$$

Obtain, with lhs the left–hand side of (5.13),

$$\begin{aligned}
lhs &= \frac{(2n)!J}{2^{2n}(n!)^2} \\
&= \frac{(2n)!2^{2n+1}}{2^{2n}(2n+1)!} \\
&= \frac{2}{2n+1}
\end{aligned}$$

This completes the proof of (5.13).

Proof of Lemma 5.5: Using the substitution

$$x^2 = t, \quad 2x\,dx = dt \ ,$$

one obtains that

$$\begin{aligned}
\int_0^\infty x^{2p-1}e^{-x^2}\, dx &= \frac{1}{2}\int_0^\infty t^{p-1}e^{-t}\, dt \\
&= \frac{1}{2}\Gamma(p)
\end{aligned}$$

We will evaluate the integral

$$I = \int_0^\infty \int_0^\infty x^{2p-1} y^{2q-1} e^{-x^2-y^2}\, dxdy$$

in two ways: (a) using Fubini's theorem, (b) using polar coordinates. Obtain (a):

$$\begin{aligned} I &= \Big(\int_0^\infty x^{2p-1} e^{-x^2}\, dx\Big)\Big(\int_0^\infty y^{2q-1} e^{-y^2}\, dy\Big) \\ &= \frac{1}{4}\,\Gamma(p)\Gamma(q) \end{aligned}$$

Also, (b), using $x = r\cos\phi, y = r\sin\phi$,

$$\begin{aligned} I &= \int_{\phi=0}^{\pi/2} \int_{r=0}^{\infty} r^{2p+2q-2} (\cos^{2p-1}\phi)\,(\sin^{2q-1}\phi)\, e^{-r^2}\, r\, drd\phi \\ &= \Big(\int_0^\infty r^{2p+2q-1} e^{-r^2}\, dr\Big)\Big(\int_0^{\pi/2} \cos^{2p-1}\phi\, \sin^{2q-1}\phi\, d\phi\Big) \\ &=: \frac{1}{2}\,\Gamma(p+q) I_1 \end{aligned}$$

To evaluate I_1 we will use the substitution

$$t = \cos^2\phi, \quad dt = -2\sin\phi\, \cos\phi\, d\phi\ .$$

We have

$$\begin{aligned} I_1 &= \frac{1}{2}\int_0^{\pi/2} (\cos^{2p-2}\phi)\,(\sin^{2q-2}\phi\,) 2\sin\phi\, \cos\phi\, d\phi \\ &= \frac{1}{2}\int_0^{\pi/2} (\cos^{2p-2}\phi)\,(1-\cos^2\phi)^{q-1}\, 2\sin\phi\, \cos\phi\, d\phi \\ &= \frac{1}{2}\int_0^{1} t^{p-1}(1-t)^{q-1}\, dt \\ &= \frac{1}{2}\, B(p,q) \end{aligned}$$

We have shown that

$$I = \frac{1}{4}\,\Gamma(p)\Gamma(q) = \frac{1}{4}\,\Gamma(p+q)B(p,q)\ ,$$

which proves the lemma. ◇

5.10 Solution of the Associated Legendre Equation

The equation reads

$$(1-x^2)y'' - 2xy' + \Big(n(n+1) - \frac{m^2}{1-x^2}\Big)y = 0 \qquad (5.15)$$

Here n and m are integers. It is remarkable that a solution $P(x) = P_n(x)$ of the equation for $m = 0$ leads, in a simple way, to a nontrivial solution for any integer m with $1 \le m \le n$.

Lemma 5.6 *Let $1 \le m \le n$ with integers m and n. The function*

$$y(x) = (1-x^2)^{m/2} D^m P(x), \quad -1 < x < 1 \ ,$$

solves (5.15) if $P(x)$ solves Legendre's equation,

$$(1-x^2)\, P''(x) - 2x\, P'(x) + n(n+1)P(x) = 0 \ . \qquad (5.16)$$

Proof: Let $u = D^m P$. We derive an equation satisfied by u by differentiating (5.16) m times:

We have

$$(1-x^2)D^{m+2}P - 2xmD^{m+1}P - m(m-1)D^mP - 2xD^{m+1}P - 2mD^mP + n(n+1)D^mP = 0 \ .$$

Collecting terms we obtain

$$(1-x^2)u'' - 2x(m+1)u' + (n^2+n-m^2-m)u = 0 \ .$$

We have

$$\begin{aligned} u &= (1-x^2)^{-m/2}y \\ u' &= mx(1-x^2)^{-\frac{m}{2}-1}y + (1-x^2)^{-m/2}y' \\ u'' &= \Big(m(1-x^2)^{-\frac{m}{2}-1} + m(m+2)x^2(1-x^2)^{-\frac{m}{2}-2}\Big)y \\ &\quad + 2mx(1-x^2)^{-\frac{m}{2}-1}y' + (1-x^2)^{-m/2}y'' \end{aligned}$$

Substitute these expressions for u, u', u'' into the equation for u and multiply by $(1-x^2)^{m/2}$. Obtain

$$(1-x^2)y'' + Q_1 y' + Q_2 y = 0$$

where

$$Q_1 = (1-x^2)2mx(1-x^2)^{-1} - 2x(m+1) = -2x$$

and

$$
\begin{aligned}
Q_2 &= n^2 + n - m^2 - m + (-2x)(m+1)mx(1-x^2)^{-1} \\
&\quad + (1-x^2)\Big(m(1-x^2)^{-1} + m(m+2)x^2(1\text{‘} - x^2)^{-2}\Big) \\
&= n^2 + n - m^2 - m + m + (1-x^2)^{-1}\Big(-2x^2m(m+1) + m(m+2)x^2\Big) \\
&= n^2 + n - m^2 + \frac{x^2}{1-x^2}\,(m^2 + 2m - 2m^2 - 2m) \\
&= n^2 + n - m^2 - \frac{m^2x^2}{1-x^2} \\
&= n^2 + n - \frac{m^2}{1-x^2}(1 - x^2 + x^2) \\
&= n^2 + n - \frac{m^2}{1-x^2}
\end{aligned}
$$

This proves the lemma. ⋄

If $P = P_n$ is the n–th Legendre polynomial, then the function $y(x)$ defined in the previous lemma is nontrivial for $1 \le m \le n$. In fact, if m is even, then y is a polynomial of degree n. If m is odd, then y is a polynomial of degree $n-1$ multiplied by $\sqrt{1-x^2}$.

For our discussion of spherical harmonic below, it will be convenient to introduce the following functions:

$$
P_n^m(x) = \frac{(-1)^m}{2^n\, n!}(1-x^2)^{m/2} D^{m+n} X^n, \quad X = x^2 - 1\ , \tag{5.17}
$$

for $-n \le m \le n$. The functions $P_n^m(x)$ are called associated Legendre functions of order m. Since

$$
P_n(x) = \frac{1}{2^n\, n!}\, D^n X^n, \quad X = x^2 - 1
$$

is the Legendre polynomial of degree n, we have

$$
P_n^m(x) = (-1)^m (1-x^2)^{m/2} D^m P_n(x) \quad \text{for} \quad 0 \le m \le n\ .
$$

However, the formula (5.17) makes sense also for $-n \le m \le -1$.

Example:

$$
\begin{aligned}
P_4^4(x) &= \frac{1}{2^4\, 4!}\,(1-x^2)^2 D^{4+4}\Big((x^2-1)^4\Big) \\
&= \frac{8!}{2^4\, 4!}\,(1-x^2)^2 \\
&= 105\,(1-x^2)^2
\end{aligned}
$$

Therefore,

$$
P_4^4(\cos\theta) = 105\, \sin^4\theta\ .
$$

In general, if $x = \cos\theta$, then $(1-x^2)^{m/2} = \sin^m\theta$. Since $D^{m+n}X^n$ is a polynomial in x, the function $P_n^m(\cos\theta)$ is a polynomial in $\sin\theta$ and $\cos\theta$ if $0 \le m \le n$:

$$P_n^m(\cos\theta) = \frac{(-1)^m}{2^n\, n!} \sin^m\theta \left(\frac{d}{dx}\right)^{m+n} (x^2-1)^n\Big|_{x=\cos\theta} \; .$$

It turns out that for $-n \le m \le -1$, the function $P_n^m(\theta)$ is also a polynomial in $\sin\theta$ and $\cos\theta$. This follows from the next lemma.

We claim that, for $1 \le m \le n$, the function $P_n^{-m}(x)$ is a multiple of the function $P_n^m(x)$. Precisely:

Lemma 5.7 *For $1 \le m \le n$ we have*

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x), \quad -1 < x < 1 \; .$$

Proof: One must show that

$$\frac{(n-m)!}{(n+m)!} X^m D^{m+n} X^n = D^{-m+n} X^n \; .$$

Essentially, this can be shown by applying Leibniz' rule of differentiation to

$$X^n = (x+1)^n (x-1)^n \; .$$

We have

$$D^{n-m}\Big((x+1)^n(x-1)^n\Big) = \sum_{j=0}^{n-m} \binom{n-m}{j} \Big(D^j(x+1)^n\Big)\Big(D^{n-m-j}(x-1)^n\Big) \; .$$

Here

$$D^j(x+1)^n = \frac{n!}{(n-j)!}(x+1)^{n-j}$$

and

$$D^{n-m-j}(x-1)^n = \frac{n!}{(m+j)!}(x-1)^{m+j} \; .$$

One obtains:

$$D^{n-m}\Big((x+1)^n(x-1)^n\Big) = \sum_{j=0}^{n-m} c_{mnj}(x+1)^{n-j}(x-1)^{m+j}$$

with

$$c_{mnj} = \frac{(n-m)!n!n!}{j!(n-m-j)!(n-j)!(m+j)!} \; .$$

Similarly,

$$D^{n+m}\Big((x+1)^n(x-1)^n\Big) = \sum_{k=0}^{n+m} \left(\begin{array}{c} n+m \\ k \end{array} \right) \Big(D^k(x+1)^n\Big)\Big(D^{n+m-k}(x-1)^n\Big) \ .$$

Note that the term in the sum is zero unless $m \le k \le n$. Therefore, with $k = m + j$,

$$\begin{aligned} D^{n+m}\Big((x+1)^n(x-1)^n\Big) &= \sum_{k=m}^{n} \left(\begin{array}{c} n+m \\ k \end{array} \right) \Big(D^k(x+1)^n\Big)\Big(D^{n+m-k}(x-1)^n\Big) \\ &= \sum_{j=0}^{n-m} \left(\begin{array}{c} n+m \\ m+j \end{array} \right) \Big(D^{m+j}(x+1)^n\Big)\Big(D^{n-j}(x-1)^n\Big) \\ &= \sum_{j=0}^{n-m} \frac{(n+m)!n!n!}{(m+j)!(n-j)!(n-m-j)!j!} (x+1)^{n-m-j}(x-1)^j \end{aligned}$$

Therefore,

$$X^m D^{n+m}\Big((x+1)^n(x-1)^n\Big) = \sum_{j=0}^{n-m} \frac{(n+m)!n!n!}{(m+j)!(n-j)!(n-m-j)!j!} (x+1)^{n-j}(x-1)^{m+j} \ .$$

Finally,

$$\frac{(n-m)!}{(n+m)!} \cdot \frac{(n+m)!n!n!}{(m+j)!(n-j)!(n-m-j)!j!} = \frac{(n-m)!n!n!}{j!(n-m-j)!(n-j)!(m+j)!} = c_{mnj} \ .$$

The lemma is proved. ◇

From the previous two lemmas, it is clear that both functions, $P_n^{-m}(x)$ and $P_n^m(x)$, satisfy the associated Legendre equation (5.15).

The functions $P_n^m(x)$ for $-n \le m \le n$ are introduced in order to define the spherical harmonics,

$$Y_n^m(\theta,\phi) = \gamma_n^m P_n^m(\cos\theta) e^{im\phi}, \quad -\pi < \theta < \pi, \quad 0 < \pi < 2\pi \ ,$$

with

$$\gamma_n^m = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}}$$

for $-n \le m \le n$ with integers m, n. We can think of $Y_n^m(\theta,\phi)$ as a function defined on the unit sphere in $\mathbb{R}^3$. We will consider spherical harmonic in the next section.

Let us first discuss the functions $P_n^m(x)$ further.

Lemma 5.8 *We claim that*

$$\int_{-1}^{1} P_p^m(x) P_q^m(x)\, dx = 0 \quad \textit{for} \quad p \neq q \ .$$

Here we may assume $|m| \leq p < q$.

Proof: We may assume that $1 \leq m \leq p < q$. Consider

$$Int = \int_{-1}^{1} X^m (D^{m+p} X^p)(D^{m+q} X^q) dx, \quad X = x^2 - 1 \ .$$

Use integration by parts [1] to remove D^{m+q} from X^q. Note that

$$D^{m+q}(X^m D^{m+p} X^p)$$

is a sum of terms

$$(D^j X^m)(D^{m+q+m+p-j} X^p), \quad 0 \leq j \leq m+q \ .$$

If $j > 2m$ then $D^j X^m = 0$. If $j \leq 2m$ then

$$2m - j + q + p > 2p \ ,$$

and therefore $D^{2m-j+q+p} X^p = 0$. This proves the lemma. $\diamond$

Lemma 5.9 *For* $|m| \leq n$ *we have*

$$\int_{-1}^{1} P_n^m(x) P_n^m(x)\, dx = \frac{(n+m)!}{(n-m)!} \frac{2}{2n+1} \ .$$

Proof: With $c_n = \frac{1}{2^n\, n!}$ and $X = x^2 - 1$ we have

$$\begin{aligned}
\int_{-1}^{1} P_n^m(x) P_n^m(x)\, dx &= (-1)^m c_n^2 \int_{-1}^{1} X^m (D^{m+n} X^n)(D^{m+n} X^n)\, dx \\
&= (-1)^n c_n^2 \int_{-1}^{1} D^{m+n}\Big(X^m (D^{m+n} X^n)\Big) X^n\, dx \\
&=: Int
\end{aligned}$$

Apply Leibniz's rule,

$$D^{m+n}\Big(X^m (D^{m+n} X^n)\Big) = \sum_{j=0}^{m+n} \left(\begin{array}{c} m+n \\ j \end{array} \right) (D^{m+n-j} X^m)(D^{m+n+j} X^n) \ .$$

[1] On the boundary terms appearing through integration by parts: We have $Int = \int_{-1}^{1} Q(x) D^{m+q} X^q\, dx$ where $Q(x) = X^m (D^{m+p} X^p)$ is a polynomial of degree $\partial Q = 2m + 2p - m - p = m + p$. The polynomial $Q(x)$ vanishes m times at $x = \pm 1$. The boundary terms read $BT_j = \pm (D^j Q)(D^{m+q-j-1} X^q)|_{-1}^{1}$. For $0 \leq j \leq m-1$ the term $D^j Q$ is zero. For $m \leq j \leq m+q-1$ the second term is zero.

If $n-j>m$ then the first term is zero. If $n-j<m$ then $m+j>n$ and $m+n+j>2n$. Therefore, the second term is zero. We must only consider the term in the above sum that is obtained for $j=n-m$. The integral in Int becomes

$$\begin{pmatrix} n+m \\ n-m \end{pmatrix} \int_{-1}^{1} (D^{2m}X^m)(D^{2n}X^n)X^n\,dx\ .$$

Here $D^{2m}X^m=(2m)!$ and $D^{2n}X^n=(2n)!$. Obtain:

$$Int=(-1)^n c_n^2 (2m)!(2n)!\,\frac{(n+m)!}{(n-m)!(2m)!}\,J \tag{5.18}$$

with

$$\begin{aligned} J &= \int_{-1}^{1} X^n dx \\ &= (-1)^n \int_{-1}^{1} (1-x^2)^n dx \\ &= (-1)^n\,\frac{2^{2n+1}(n!)^2}{(2n+1)!} \end{aligned}$$

In the last equation we have used (5.14).

Together with (5.18):

$$\begin{aligned} Int &= \frac{(2m)!(2n)!(n+m)!2^{2n+1}n!n!}{2^{2n}n!n!(n-m)!(2m)!(2n+1)!} \\ &= \frac{(n+m)!}{(n-m)!}\,\frac{2}{2n+1} \end{aligned}$$

This proves the lemma. $\diamond$

5.11 Spherical Harmonics as Eigenfunctions

Let r,θ,ϕ denote the usual spherical coordinates. Recall that

$$\Delta\psi=\frac{1}{r^2}\,(r^2\psi_r)_r+\frac{1}{r^2}\Big(\psi_{\theta\theta}+\cot\theta\,\psi_\theta+\frac{1}{\sin^2\theta}\,\psi_{\phi\phi}\Big)\ .$$

If $Y(\theta,\phi)$ is a function on the unit sphere, then define

$$LY=Y_{\theta\theta}+\cot\theta\,Y_\theta+\frac{1}{\sin^2\theta}Y_{\phi\phi}\ .$$

With this notation, we can write the equation $\Delta\psi=0$ as

$$(r^2\psi_r)_r+L\psi=0\ . \tag{5.19}$$

We try to find solutions of Laplace's equation (5.19) of the form

$$\psi(r,\theta,\phi)=R(r)Y(\theta,\phi)\ .$$

Substitution into (5.19) and division by ψ yields

$$\frac{1}{R}(r^2R')' + \frac{1}{Y}LY(\theta,\phi) = 0 \ .$$

Denoting the separation constant by Q we obtain

$$\begin{aligned} (r^2R')' - QR &= 0 \\ -LY &= QY \end{aligned}$$

One can prove that the operator $-L$ has the eigenvalues $Q_n = n(n+1)$ where $n = 0, 1, 2, \ldots$. Each eigenvalue $n(n+1)$ has multiplicity $2n+1$. We will construct an orthonormal basis of eigenfunctions for L,

$$Y_n^m(\theta,\phi) = \gamma_n^m P_n^m(\cos\theta)e^{im\phi}, \quad -n \le m \le n \ ,$$

where γ_n^m is defined as above,

$$\gamma_n^m = \sqrt{\frac{2n+1}{4\pi}\,\frac{(n-m)!}{(n+m)!}} \ .$$

With $Q = n(n+1)$, the R–equation is an Euler equation with general solution

$$R(r) = c_1 r^n + \frac{c_2}{r^{n+1}} \ .$$

Assume $Q = n(n+1)$. Substitute

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$$

into

$$Y_{\theta\theta} + \cot\theta\, Y_\theta + \frac{1}{\sin^2\theta}Y_{\phi\phi} + n(n+1)Y = 0 \ .$$

Divide by Y. Obtain

$$\frac{1}{\Theta}\Big(\Theta'' + \cot\theta\,\Theta'\Big) + \frac{1}{\sin^2\theta}\,\frac{\Phi''}{\Phi} + n(n+1) = 0 \ .$$

With $\Phi''/\Phi = -m^2$ obtain

$$\Theta'' + \cot\theta\,\Theta' + \Big(n(n+1) - \frac{m^2}{s^2}\Big)\Theta = 0 \ .$$

Let $\Theta(\theta) = P(\cos\theta)$,

$$\Theta' = -\sin\theta\, P', \quad \Theta'' = -\cos\theta\, P' + \sin^2\theta P'' \ .$$

Obtain

$$\sin^2\theta\, P'' - 2\cos\theta\, P' + \Big(n(n+1) - \frac{m^2}{\sin^2\theta}\Big)P = 0 \ .$$

If $x = \cos\theta$, then $\sin^2\theta = 1 - x^2$, thus

$$(1-x^2)P'' - 2xP' + \Big(n(n+1) - \frac{m^2}{1-x^2}\Big)P = 0 \ .$$

The functions $P_n^m(x)$ satisfy this equation for $-n \le m \le n$. Then

$$Z_n^m(\theta,\phi) := P_n^m(\cos\theta)e^{im\phi}$$

satisfies

$$-LZ_n^m = n(n+1)Z_n^m \ .$$

We claim that the system of functions

$$Y_n^m(\theta,\phi), \quad -n \le m \le n, \quad n = 0, 1, \ldots$$

is an orthonormal system in $L_2(\mathcal{S})$ where $\mathcal{S}$ is the unit sphere. Recall that the element of area for the unit sphere $\mathcal{S}$ is

$$dS = \sin\theta \, d\theta d\phi \ .$$

Integration over the sphere $\mathcal{S}$:

$$\int_{\mathcal{S}} Z \, dS = \int_0^{2\pi}\int_0^{\pi} Z(\theta,\phi)\sin\theta \, d\theta d\phi$$

The L_2–inner–product of two functions defined on $\mathcal{S}$ is:

$$(Z_1, Z_2) = \int_{\mathcal{S}} \bar{Z}_1 Z_2 \, dS$$

The L_2–norm is:

$$\|Z\| = (Z,Z)^{1/2}$$

Orthogonality: If $m_1 \neq m_2$ then

$$\int_0^{2\pi} e^{-im_1\phi}e^{im_2\phi}\, d\phi = 0 \ .$$

If $n_1 \neq n_2$ then

$$\begin{aligned}\int_0^{\pi} P_{n_1}^m(\cos\theta)P_{n_2}^m(\cos\theta)\sin\theta\, d\theta &= \int_{-1}^{1} P_{n_1}^m(x)P_{n_2}^m(x)\, dx \\ &= 0\end{aligned}$$

Therefore, if $m_1 \neq m_2$ or $n_1 \neq n_2$ then

$$\Big(Z_{n_1}^{m_1}, Z_{n_2}^{m_2}\Big) = 0 \ .$$

Normalization:

$$\begin{aligned}
\|Z_n^m\|^2 &= 2\pi \int_{-1}^{1} (P_n^m)^2(x)\, dx \\
&= \frac{4\pi}{2n+1} \frac{(n+m)!}{(n-m)!} \\
&= (1/\gamma_n^m)^2
\end{aligned}$$

Therefore, the functions

$$Y_n^m(\theta,\phi), \quad -n \le m \le n, \quad n = 0, 1, \ldots$$

form an orthonormal system in $L_2(\mathcal{S})$. One can prove that this systen is complete, i.e., if $y = y(\theta,\phi)$ is any function in $L_2(\mathcal{S})$ then the series

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn} Y_n^m(\theta,\phi) \quad \text{with} \quad a_{mn} = (Y_n^m, y)$$

converges to y w.r.t. the L_2–norm on $\mathcal{S}$.

5.12 Spherical Harmonics are Restrictions of Harmonic Polynomials to the Unit Sphere

We have

$$Y_n^m(\theta,\phi) = \gamma_n^m \, P_n^m(\cos\theta) e^{im\phi}$$

where

$$P_n^m(x) = \frac{1}{2^n n!} (1-x^2)^{m/2} D^{m+n} X^n, \quad X = x^2 - 1 \ .$$

Set

$$\psi(r,\theta,\phi) = r^n Y_n^m(\theta,\phi)$$

to obtain a function defined in all space.

Claim:

$$\Delta\psi = 0$$

Proof: We have

$$\Delta\psi = \frac{1}{r^2} (r^2\psi_r)_r + \frac{1}{r^2} L\psi$$

Here

$$L(r^n Y_n^m) = -r^n n(n+1) Y_n^m$$

Also,

$$
\begin{aligned}
(r^n)_r &= n r^{n-1} \\
r^2 (r^n)_r &= n r^{n+1} \\
(r^2 \psi_r)_r &= n(n+1) r^n Y_n^m
\end{aligned}
$$

This shows that $\Delta\psi = 0$.

Claim: If one writes $\psi = \psi(r, \theta, \phi)$ in Cartesian coordinates x, y, z, then one obtains a polynomial in x, y, z which is homogeneous of degree n.

Proof: It suffices to prove this for $0 \le m \le n$. We have

$$\psi = c r^n \sin^m \theta \, (D^{m+n} X^n)|_{x=\cos\theta} \, e^{im\phi}$$

Here

$$e^{im\phi} = (\cos\phi + i \sin\phi)^m$$

is a sum of terms

$$\cos^l \phi \, \sin^{m-l} \phi, \quad 0 \le l \le m \ .$$

Also,

$$X^n = (x^2 - 1)^n = \sum_{j=0}^{n} c_{jn} x^{2n-2j} = x^{2n} + c_{1n} x^{2n-2} + \dots \ .$$

Therefore, $D^{m+n} X^n$ is a sum of terms

$$x^{n-m-2j}, \quad n - m \ge n - m - 2j \ge 0 \ .$$

It follows that ψ is a sum of terms

$$H = r^n \sin^m \theta \cos^{n-m-2j} \theta \cos^l \phi \sin^{m-l} \phi$$

where

$$0 \le l \le m \quad \text{and} \quad n - m - 2j \ge 0 \ .$$

Recall that

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
H &= r^n \sin^l \theta \cos^l \phi \sin^{m-l} \theta \sin^{m-l} \phi \cos^{n-m-2j} \theta \\
&= x^l y^{m-l} r^{n-m} \cos^{n-m-2j} \theta \\
&= x^l y^{m-l} r^{2j} z^{n-m-2j} \\
&= x^l y^{m-l} (x^2 + y^2 + z^2)^j z^{n-m-2j}
\end{aligned}
$$

This is a polynomial which is homogeneous of degree n.

2D Analogy. Consider

$$p(x,y) = (x+iy)^n, \quad q(x,y) = (x-iy)^n \ .$$

It is easy to see that $p(x,y)$ and $q(x,y)$ are polynomials which are homogeneous of degree n. The corresponding restrictions to the unit circle are

$$p(\cos\phi, \sin\phi) = e^{in\phi}$$
$$q(\cos\phi, \sin\phi) = e^{-in\phi}$$

The functions $e^{\pm in\phi}$ are the 'spherical harmonics' on the unit circle in 2D. They are restrictions to the unit circle of the homogeneous polynomials p and q.

3D Generalization. The functions $Y_n^m(\theta,\phi)$, defined on the unit sphere in $\mathbb{R}^3$, are the generalizations of the functions $e^{\pm in\phi}$ defined on the unit circle in $\mathbb{R}^2$. The expansion in terms of the spherical harmonics $Y_n^m(\theta,\phi)$ generalizes the Fourier expansion in terms of the functions $e^{\pm in\phi}$.

6 Second Order Linear Homogeneous ODEs

6.1 Ordinary Points, Regular Singular Points, Irregular Singular Points

Consider the 2nd order differential equation

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0 \tag{6.1}$$

where z is a complex variable. We assume that the functions $p(z)$ and $q(z)$ are holomorphic in

$$0 < |z - z_0| < R \ .$$

In other words, the point z_0 is an isolated singularity of $p(z)$ and $q(z)$.

Case 1: The functions $p(z)$ and $q(z)$ have a removable singularity at z_0. Then z_0 is called an ordinary point for (6.1).

Case 2: Either $p(z)$ or $q(z)$ has a singularity at z_0 that is not removable. Then z_0 is called a singular point for (6.1).

Case 2a: Suppose we have Case 2. Assume that $p(z)$ has a removable singularity or a first order pole at z_0 and $q(z)$ has a removable singularity or a first order pole or a second order pole at z_0. Under these assumptions, the point z_0 is called a regular singular point for (6.1).

Case 2b: In all other cases, z_0 is called an irregular singular point for (6.1).

Examples for equations: $y''(z) = zy(z)$ is Airy's equation. All points are ordinary points.

For Bessel's equation

$$z^2 y'' + zy' + (z^2 - m^2)y = 0$$

the point $z_0 = 0$ is a regular singular point.

For the equation

$$z^3 y'' + zy' - 2y = 0$$

the point $z_0 = 0$ is an irregular singular point. Note that the function

$$y = e^{1/z}$$

solves this equation and has an essential singularity at $z = 0$.

We will show the following: Solutions near an ordinary point are analytic. Solutions near a regular singular point z_0 can be singular at z_0 but the singularity is of a very definite nature. In particular, it cannot be an essential singularity.

6.2 Equations with Constant Coefficients

Solve

$$y'' + ay' + by = 0$$

using $y = e^{rz}$. Obtain the characteristic equation

$$r^2 + ar + b = 0 \ .$$

If the roots are $r_{1,2}$ with $r_1 \neq r_2$ then the general solution is

$$y(z) = c_1 e^{r_1 z} + c_2 e^{r_2 z} \ .$$

Consider the case $r_1 = r_2$ (which occurs for $a^2 = 4b$) by perturbing the equation:

$$y'' + ay' + (\frac{a^2}{4} - \varepsilon^2)y = 0 \ .$$

The roots are

$$r_{1,2} = -\frac{a}{2} \pm \varepsilon \ .$$

The corresponding solutions are

$$y_{1,\varepsilon} = e^{-az/2} e^{\varepsilon z}, \quad y_{2,\varepsilon} = e^{-az/2} e^{-\varepsilon z} \ .$$

As $\varepsilon \to 0$, both solutions approach

$$y_1(z) = e^{-az/2} \ .$$

Consider the solution

$$\begin{aligned}
y_{3,\varepsilon}(z) &= \frac{1}{2\varepsilon}\Big(y_{1,\varepsilon}(z) - y_{2,\varepsilon}(z)\Big) \\
&= e^{-az/2}\frac{1}{2\varepsilon}\Big(1 + \varepsilon z - (1 - \varepsilon z) + \mathcal{O}(\varepsilon^2 z^2)\Big) \\
&= ze^{-az/2} + \mathcal{O}(\varepsilon z^2)
\end{aligned}$$

As $\varepsilon \to 0$, this solution approaches

$$y_3(z) = ze^{-az/2} \ .$$

The two solutions

$$y_1(z) = e^{-az/2} \quad \text{and} \quad y_3(z) = ze^{-az/2}$$

form a fundamental set for the equation

$$y'' + ay' + \frac{a^2}{4}y = 0 \ .$$

6.3 Cauchy–Euler Equations

The equation has the form

$$z^2y''(z) + azy'(z) + by(z) = 0 \ . \tag{6.2}$$

One solves it by $y = z^r$. Then r must satisfy the indicial equation

$$r(r-1) + ar + b = 0 \ .$$

If $r_1 \neq r_2$ then

$$z^{r_1}, \quad z^{r_2}$$

form a fundamental set. For $r_1 = r_2$ obtain

$$y_1 = z^{r_1}, \quad y_2 = z^{r_1} \log z \ .$$

Example 6.1: The differential equation

$$2z^2y'' + 3zy' - y = 0$$

has the indicial equation

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = 0 \ .$$

The roots are

$$r_1 = \frac{1}{2}, \quad r_2 = -1 \ .$$

The general solution of the differential equation is

$$y(z) = c_1\sqrt{z} + \frac{c_2}{z} \ .$$

Remark: The Cauchy–Euler equation (6.2) can be transformed to an equation with constant coefficients as follows: Let

$$z = e^t, \quad \ln z = t, \quad y(z) = u(t) = u(\ln z) \ .$$

Obtain:

$$\begin{aligned} y'(z) &= \frac{1}{z}\, u'(\ln z) \\ y''(z) &= -\frac{1}{z^2}\, u'(\ln z) + \frac{1}{z^2}\, u''(\ln z) \end{aligned}$$

Equation (6.2) becomes

$$u''(t) + (a-1)u'(t) + bu(t) = 0 \ .$$

The characteristics equation for the u–equation is

$$r^2 + (a-1)r + b = 0 \ .$$

This equation agrees with the indicial equation for (6.2). If $r_1 = r_2 = r$ is a double root, then the solution

$$u_2(t) = te^{rt}$$

transforms to

$$y_2(z) = (\ln z)z^r \ .$$

6.4 Series Solutions Near an Ordinary Point

Consider

$$y'' + p(z)y' + q(z)y = 0$$

where

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \quad q(z) = \sum_{n=0}^{\infty} q_n z^n$$

for $|z| < R_0$. Using the Liouville transformation

$$y = e^{-P/2}w \ ,$$

where $P' = p$, one obtains an equation for w,

$$w'' + Qw = 0, \quad Q = q - \frac{1}{2}p' - \frac{1}{4}p^2 \ .$$

Thus we may assume $p = 0$.

Details: If $p(z) = \sum_{n=0}^{\infty} p_n z^n$ conerges for $|z| < R_0$, then $P(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} p_n z^{n+1}$ also converges for $|z| < R_0$ and $P'(z) = p(z)$. Define the function $w(z)$ by

$$y(z) = e^{-P(z)/2}w(z) \ .$$

Obtain:

$$\begin{aligned} y' &= e^{-P/2}w' - \frac{p}{2}e^{-P/2}w \\ y'' &= e^{-P/2}w'' - pe^{-P/2}w' - \frac{p'}{2}e^{-P/2}w + \frac{p^2}{4}e^{-P/2}w \end{aligned}$$

Substituting the above expressions for y, y', y'' into the equation

$$y'' + py' + qy = 0$$

yields

$$\begin{aligned} 0 &= w'' - pw' + (\frac{p^2}{4} - \frac{p'}{2})w + pw' - \frac{p^2}{2}w + qw \\ &= w'' + (q - \frac{p'}{2} - \frac{p^2}{4})w \end{aligned}$$

Solution of $y''(z) + q(z)y(z) = 0$. We first proceed formally. Let

$$y(z) = \sum_{j=0}^{\infty} a_j z^j \ .$$

Differentiate twice:

$$y''(z) = \sum_{j=0}^{\infty} a_{j+2}(j+2)(j+1)z^j \ .$$

Also,

$$q(z)y(z) = \sum_{j=0}^{\infty} b_j z^j, \quad b_j = \sum_{i=0}^{j} q_i a_{j-i} \ .$$

Obtain: a_0, a_1 are free. Then, for $j = 0, 1, \ldots$

$$-a_{j+2}(j+2)(j+1) = \sum_{i=0}^{j} q_i a_{j-i} \ .$$

This determines the a_j recursively once a_0, a_1 are chosen.

Theorem 6.1 *If the series $\sum_{j=0}^{\infty} q_j z^j$ converges for $|z| < R_0$, then the series $\sum_{j=0}^{\infty} a_j z^j$ also converges for $|z| < R_0$.*

Proof: Let $|z| < R < R_0$. Cauchy's inequalities imply

$$|q_j| \le \frac{B}{R^j}, \quad j = 0, 1, \ldots$$

with

$$B = \max_{|z|=R} |q(z)| \ .$$

Claim: There exists $A > 0$ with

$$|a_j| \le \frac{A}{R^j}, \quad j = 0, 1, \ldots$$

Once this is shown, convergence of the series for $y(z)$ follows from

$$|a_j||z|^j \le A\Big(\frac{|z|}{R}\Big)^j, \quad |z|/R < 1 \ ,$$

by the comparison test.

First fix some n and assume that

$$|a_j| \le \frac{A}{R^j}, \quad 0 \le j \le n \ .$$

Obtain:

$$\begin{aligned}
|a_{n+1}| &\le \frac{1}{(n+1)n} \sum_{i=0}^{n-1} |q_i a_{n-1-i}| \\
&\le \frac{1}{(n+1)n} \sum_{i=0}^{n-1} \frac{B}{R^i} \frac{A}{R^{n-1-i}} \\
&= \frac{1}{n+1} \frac{AB}{R^{n-1}} \\
&= \frac{A}{R^{n+1}} \cdot \frac{BR^2}{n+1}
\end{aligned}$$

Choose n so large that $BR^2/(n+1) \le 1$. Then choose A so large $|a_j| \le A/R^j$ for $j = 0, \ldots, n$. Under these assumptions, the above estimate shows inductively that $|a_j| \le A/R^j$ for all j. This completes the proof. $\diamond$

6.5 Solutions Near a Regular Singular Point

Consider

$$z^2 y'' + zp(z)y' + q(z)y = 0$$

where

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \quad q(z) = \sum_{n=0}^{\infty} q_n z^n$$

for $|z| < R_0$. The corresponding Euler equation is

$$z^2 y'' + zp_0 y' + q_0 y = 0 \ .$$

The indicial equation is

$$r(r-1) + p_0 r + q_0 = 0 \ .$$

Denote the solutions by $r_{1,2}$ where

$$\mathrm{Re}\, r_1 \ge \mathrm{Re}\, r_2 \ .$$

The basic idea is to find solutions of the form

$$y_1(z) = z^{r_1} \sum_{n=0}^{\infty} a_n z^n$$

and

$$y_2(z) = z^{r_2} \sum_{n=0}^{\infty} b_n z^n \ .$$

The construction of y_1 is always possible. The construction of y_2 may fail.

For y_1 one obtains the following: The coefficient a_0 is free. Then, for $n \geq 1$, let

$$A_n = n(n + r_1 - r_2) \ .$$

Note that $|A_n| \geq n^2$. Obtain the conditions

$$A_n a_n = -\sum_{k=0}^{n-1} (k + r_1) p_{n-k} a_k - \sum_{k=0}^{n-1} q_{n-k} a_k \ .$$

This determines the a_n uniquely once a_0 is chosen.

Claim: The series for $y_1(z)$ converges for $|z| < R_0$.

Proof: Let $|z| < R < R_0$. First assume that for some $A > 0, m > 0, n > 0$ we have

$$|a_j| \leq AR^{-j}(j+1)^m \quad 0 \leq j \leq n-1 \ .$$

The induction step can be completed for m, n large enough. The induction hypothesis then holds for A large.

As an auxiliary result we use that

$$\sum_{i=1}^{n-1} i^m \leq \int_1^n x^m \, dx \leq \frac{n^{m+1}}{m+1} \ .$$

If one tries to determine $y_2(z)$ in the same way, then the recursion involves

$$B_n = n(n + r_2 - r_1)$$

instead of A_n. If

$$B_n \neq 0$$

for every n, then there is $\delta > 0$ with

$$|B_n| \geq \delta n^2 \ .$$

Assume $B_n \neq 0, n = 1, 2, \ldots$ Once b_0 is chosen, the b_n are uniquely determined. Convergence of the series for $y_2(z)$ follows in the same way as for $y_1(z)$.

If $B_n = 0$ for some n, then, in general, the construction of y_2 will break down. We will discuss this case for Bessel's equation.

6.6 Bessel Functions of the First Kind

Note: Using analytic continuation, the gamma function $\Gamma(q)$ is defined for all complex q except for

$$q \in \{0, -1, -2, \ldots\} \ .$$

The function $1/\Gamma(q)$ is defined and analytic for all complex q and is zero for

$$q \in \{0, -1, -2, \ldots\} \ .$$

Consider

$$z^2 y'' + z y' + (z^2 - m^2) y = 0 \ . \tag{6.3}$$

We may assume $\mathrm{Re}\, m \geq 0$. The indicial equation is

$$r^2 - m^2 = 0$$

with roots

$$r_1 = m, \quad r_2 = -m \ .$$

As in the general case, let

$$y_1 = z^m \sum_{n=0}^{\infty} a_n z^n \ .$$

Obtain

$$A_n = n(n + 2m) \ .$$

The conditions for the a_n become:

$$\begin{aligned} 0 a_0 &= 0 \\ A_1 a_1 &= 0 \\ A_n a_n &= -a_{n-2} \end{aligned}$$

Thus $a_1 = 0$ and then $a_n = 0$ for all odd n. By convention,

$$a_0 = \frac{1}{2^m \Gamma(m+1)} \ .$$

Then one obtains

$$y_1(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+m+1)} \left(\frac{z}{2}\right)^{2n} =: J_m(z) \ .$$

When constructing $y_2(z)$ one must consider

$$B_n = n(n - 2m) \ .$$

The construction of the second solution works if $B_n \neq 0$ for every $n = 1, 2,$ Choosing

$$b_0 = \frac{1}{2^{-m}\Gamma(-m+1)}$$

one then obtains

$$y_2(z) = \left(\frac{z}{2}\right)^{-m} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-m+1)} \left(\frac{z}{2}\right)^{2n} =: J_{-m}(z) \ .$$

The general definition of $J_\nu(z)$ is

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n} \ .$$

If one uses the above convention for $1/\Gamma(q)$, then $J_\nu(z)$ is defined for all $\nu \in \mathbb{C}$. If

$$m \notin \{0, 1, 2, \ldots\}$$

then J_m, J_{-m} form a fundamental set for (6.3).

Remark: If $m = g + \frac{1}{2}$ is a half–integer, then $B_n = 0$ occurs for an odd number n. The above definition for $y_2(z)$ still follows the general process if one chooses $b_n = 0$ for all odd n. In fact, $B_n = 0$ occurs for $n = 2g + 1 =: n_0$. One only needs $b_n = 0$ for odd $n < n_0$. Then b_{n_0} is free. However, the corresponding solution becomes a multiple of $J_m(z)$.

Consider $\nu = -3$, for example. It is easy to show that

$$J_{-3}(z) = -J_3(z) \ ,$$

and, more generally,

$$J_{-m}(z) = (-1)^m J_m(z), \quad m = 1, 2, \ldots$$

Thus, if $m \in \{0, 1, 2, \ldots\}$, then we do not yet have a fundamental set for (6.3).

6.7 Bessel Functions of the Second Kind

For $\nu \in \mathbb{C} \setminus \mathbb{Z}$ define

$$N_\nu(z) = \frac{\cos(\nu\pi)J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)} \ .$$

The function N_ν is called a Neumann function or Bessel function of the second kind. For integer m define

$$N_m(z) = \lim_{\nu \to m} N_\nu(z) \ .$$

One can prove that the pair J_m, N_m always forms a fundamental set for (6.3).

The function $N_0(z)$ for $0 < z << 1$. In $J_\varepsilon(z)$ and $J_{-\varepsilon}(z)$ we only consider the first terms, obtained for $n = 0$, since $|z| << 1$. Then we obtain

$$N_0(z) \approx \frac{1}{\varepsilon\pi}\Big[\Big(\frac{z}{2}\Big)^{\varepsilon}\frac{1}{\Gamma(1+\varepsilon)} - \Big(\frac{z}{2}\Big)^{-\varepsilon}\frac{1}{\Gamma(1-\varepsilon)}\Big] \ .$$

Set

$$\Gamma'(1) =: -\beta < 0 \ .$$

Then we have

$$\Gamma(1+\varepsilon) \approx 1 - \beta\varepsilon, \quad \Gamma(1-\varepsilon) \approx 1 + \beta\varepsilon \ .$$

Therefore,

$$\begin{aligned} N_0(z) &\approx \frac{1}{\varepsilon\pi}\Big[\Big(\frac{z}{2}\Big)^{\varepsilon} - \Big(\frac{z}{2}\Big)^{-\varepsilon}\Big] \\ &+ \frac{\beta}{\varepsilon}\Big[\Big(\frac{z}{2}\Big)^{\varepsilon} + \Big(\frac{z}{2}\Big)^{-\varepsilon}\Big] \end{aligned}$$

Note that the function $f(\varepsilon) = \alpha^{\varepsilon} = e^{\varepsilon \log \alpha}$ has the derivative

$$f'(0) = \log \alpha \ .$$

Therefore,

$$\begin{aligned} N_0(z) &\approx \frac{1}{\pi}\Big(\log(z/2) - \log(2/z)\Big) + \frac{2\beta}{\pi} \\ &= \frac{2}{\pi}\left(\log z - \log 2 + \beta\right) \end{aligned}$$

One can prove that

$$\begin{aligned} -\Gamma'(1) &= \beta \\ &= \gamma \\ &= \lim_{n\to\infty}\Big(\sum_{j=1}^{n}\frac{1}{j} - \log n\Big) \\ &\approx 0.577\ldots \end{aligned}$$

Note:

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\,dt \ ,$$

therefore,

$$\Gamma'(1) = \int_0^\infty \log t \ e^{-t}\,dt =: Int \ .$$

Write

$$Int = \int_0^1 + \int_1^\infty$$

and use integration by parts.

6.8 Recurrence Relations

It is claimed that

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{z} J_\nu(z) \tag{6.4}$$

$$J_{\nu-1} - J_{\nu+1} = 2J'_\nu(z) \tag{6.5}$$

We show (6.4) ...

Recall that

$$\begin{aligned} J_{1/2} &= \Big(\frac{2}{\pi z}\Big)^{1/2} \sin z \\ J_{-1/2} &= \Big(\frac{2}{\pi z}\Big)^{1/2} \cos z \end{aligned}$$

Take $\nu = \frac{1}{2}$ in (6.4). Obtain that

$$J_{-1/2} + J_{3/2} = \frac{1}{z} J_{1/2} \ .$$

Thus, $J_{3/2}$ is a combination of $\sin z, \cos z$, multiplied by powers of z.

More specifically,

$$\begin{aligned} J_{3/2}(z) &= \frac{1}{z} J_{1/2}(z) - J_{-1/2}(z) \\ &= \Big(\frac{2}{\pi}\Big)^{1/2} z^{3/2} \Big(\frac{\sin z}{z^3} - \frac{\cos z}{z^2}\Big) \end{aligned}$$

For $1 << z$ one obtains the approximation

$$J_{3/2}(z) \approx -\Big(\frac{2}{\pi}\Big)^{1/2} z^{-1/2} \cos z \ .$$

Remark: One can show that for any real ν:

$$J_\nu(z) = \Big(\frac{2}{\pi z}\Big)^{1/2} \Big(\cos(z - \frac{\nu\pi}{2} - \frac{\pi}{4}) + \mathcal{O}(\frac{1}{z})\Big) \quad \text{for} \quad z \to \infty \ .$$

We can also obtain an approximation of $J_{3/2}(z)$ for $0 < z << 1$ by substituting the power series for $\sin z$ and $\cos z$,

$$\begin{aligned} J_{3/2}(z) &= \Big(\frac{2}{\pi}\Big)^{1/2} z^{3/2} \Big(z^{-3}\Big(z - \frac{z^3}{6} + \ldots\Big) - z^{-2}\Big(1 - \frac{z^2}{2} + \ldots\Big)\Big) \\ &= \Big(\frac{2}{\pi}\Big)^{1/2} z^{3/2} \Big(\frac{1}{3} + \mathcal{O}(z^2)\Big) \end{aligned}$$

6.9 Liouville Transformation

Consider, for $x > 0$,

$$x^2 y'' + xy' + (x^2 - m^2)y = 0 \ .$$

Write

$$y'' + \frac{1}{x}y' \ldots$$

Thus,

$$\begin{aligned} p(x) &= \frac{1}{x} \\ P(x) &= \log x \\ e^{-P(x)/2} &= x^{-1/2} \end{aligned}$$

Introduce the new variable $w(x)$ by

$$y = x^{-1/2} w \ .$$

Obtain

$$w'' + \Big(1 - \frac{m^2 - 1/4}{x^2}\Big) w = 0$$

Take $m = -\frac{1}{2}$. If

$$J_{-1/2}(x) = x^{-1/2} w(x)$$

obtain that

$$w'' + w = 0 \ .$$

From the series expansion of $J_{-1/2}$ we also know that

$$J_{-1/2}(x) = x^{-1/2} \Big(\sqrt{\frac{2}{\pi}} + \mathcal{O}(x^2) \Big) \ .$$

Therefore,

$$w(0) = \sqrt{\frac{2}{\pi}}, \quad w'(0) = 0 \ .$$

It follows that

$$\begin{aligned} w(x) &= \sqrt{\frac{2}{\pi}} \cos x \ , \\ J_{-1/2}(x) &= \sqrt{\frac{2}{\pi x}} \cos x \ . \end{aligned}$$

The equation for w, together with Sturm's comparison theorem, can be used to obtain information about the zeros of Bessel functions.

6.10 Sturm's Comparison Theorem

We first consider two simple linear ODEs. Let $0 < \omega_1 < \omega_2$ be constants and let $y_1(x)$ and $y_2(x)$ satisfy

$$\begin{aligned} y_1'' + \omega_1^2 y_1 &= 0 \\ y_2'' + \omega_2^2 y_2 &= 0 \end{aligned}$$

We have

$$y_1(x) = A \sin(\omega_1 x + \alpha), \quad y_2(x) = B \sin(\omega_2 x + \beta) \ .$$

Let us assume that neither y_1 nor y_2 is identically zero. Then, if d_1 is the distance between two consecutive zeros of y_1 and d_2 is the distance between two consecutive zeros of y_2, we have

$$\omega_1 d_1 = \pi = \omega_2 d_2 \ .$$

In particular,

$$0 < d_2 < d_1 \ .$$

The function y_2 oscillates faster than y_1. Between any two consecutive zeros of y_1 there is at least one zero of y_2.

Sturm's comparison theorem generalizes this observation to solutions of two equations with variable coefficients.

Theorem 6.2 *Let $I = (a, b)$ denote an interval and let $g_1, g_2 \in C(I)$ denote two real functions with $g_1(x) < g_2(x)$ for all $x \in I$. Assume that $y_1, y_2 \in C^2(I)$ satisfy*

$$y_1'' + g_1 y_1 = 0 \quad \textit{and} \quad y_2'' + g_2 y_2 = 0 \quad \textit{in} \quad I \ .$$

Assume that neither y_1 nor y_2 is identically zero. Then, if

$$y_1(p) = y_1(q) = 0 \quad \textit{and} \quad p < q \ ,$$

there is r with

$$y_2(r) = 0, \quad p < r < q \ .$$

In other words, the function y_2 has a zero between any two zeros of y_1.

Prrof: We may assume that p and q are two consecutive zeros of y_1 and

$$y_1(x) > 0 \quad \text{for} \quad p < x < q \ .$$

If a zero r of y_2 with $p < r < q$ does not exist, then we may assume that

$$y_2(x) > 0 \quad \text{for} \quad p < x < q \ .$$

From

$$\begin{aligned} y_2 y_1'' + g_1 y_1 y_2 &= 0 \\ y_1 y_2'' + g_2 y_1 y_2 &= 0 \end{aligned}$$

we obtain

$$y_2 y_1'' - y_1 y_2'' = (g_2 - g_1) y_1 y_2 \ .$$

The right side is strictly positive for $p < x < q$. Also, through integration by parts,

$$\int_p^q y_2 y_1'' \, dx = y_2 y_1' \Big|_p^q - \int_p^q y_2' y_1' \, dx$$

and

$$\int_p^q y_1 y_2'' \, dx = y_1 y_2' \Big|_p^q - \int_p^q y_1' y_2' \, dx \ .$$

In the second case, the boundary term is zero since $y_1(p) = y_1(q) = 0$. Therefore,

$$\int_p^q (y_2 y_1'' - y_1 y_2'') \, dx = (y_2 y_1') \Big|_p^q = y_2(q) y_1'(q) - y_2(p) y_1'(p) \ .$$

Here we have $y_2(p) \geq 0, y_2(q) \geq 0$ and $y_1'(p) > 0 > y_1'(q)$. This implies that

$$\int_p^q (y_2 y_1'' - y_1 y_2'') \, dx \leq 0 \ .$$

Since $(g_2 - g_1) y_1 y_2$ is strictly positive for $p < x < q$, one has obtained a contradiction. This proves the theorem.