Mathematical Methods in Science and Engineering
Part III
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Contents

7 Applications and Properties of Bessel Functions

Let $a > 0$ and let

$$B_a = \{(x,y) \ : \ x^2 + y^2 < a^2\}$$

denote the open disk of radius a centered at zero. We will use polar coordinates ρ, ϕ in the plane.

Consider the wave equation for a function $u(\rho, \phi, t)$,

$$u_{tt} = c^2 \Delta u \quad \text{in} \quad B_a \times [0, \infty)$$

under the boundary condition

$$u = 0 \quad \text{on} \quad \partial B_a \times [0, \infty)$$

and the initial condition

$$u(\rho, \phi, 0) = f(\rho, \phi), \quad u_t(\rho, \phi, 0) = g(\rho, \phi)$$

where f and g are given smooth functions that are compatible with the boundary condition.

Recall that, in polar coordinates,

$$u_{tt} = c^2 \Big(u_{\rho\rho} + \frac{1}{\rho}\, u_\rho + \frac{1}{\rho^2}\, u_{\phi\phi} \Big) \ .$$

In the next section we consider the special case where

$$f = f(\rho), \quad g = g(\rho)$$

do not depend in ϕ.

7.1 The 2D Wave Equation: The Case of Circular Symmetry

Under the assumption that f and g do not depend on ϕ we expect that u will also be independent of ϕ. First, ignoring initial and boundary conditions, we seek solutions $u(\rho, t)$ of the wave equation in separated variables:

$$u(\rho, t) = \alpha(t) R(\rho)$$

Obtain

$$\alpha''(t) R(\rho) = c^2 \alpha(t) \Big(R''(\rho) + \frac{1}{\rho}\, R'(\rho) \Big)$$

or

$$\frac{\alpha''(t)}{c^2 \alpha(t)} = \frac{1}{R(\rho)}\, (R''(\rho) + \frac{1}{\rho} R'(\rho)) =: -k^2 \ .$$

We choose the separation constant to be $-k^2 \le 0$ since we expect the solution to be oscillatory in time.

Obtain:

$$\begin{aligned} \alpha''(t) + c^2 k^2 \alpha(t) &= 0 \\ R''(\rho) + \frac{1}{\rho} R'(\rho) + k^2 R(\rho) &= 0 \end{aligned}$$

The radial equation can be rewritten as

$$\rho^2 R''(\rho) + \rho R'(\rho) + k^2 \rho^2 R(\rho) = 0 \ .$$

It requires the boundary conditions

$$R(0) \quad \text{finite}, \quad R(a) = 0 \ .$$

Let

$$R(\rho) = y(k\rho)$$

where $y = y(z)$ is a new unknown function. Then the above equation for $R(\rho)$ transforms to

$$z^2 y''(z) + z y'(z) + z^2 y(z) = 0 \ ,$$

which is Bessel's equation of order zero. The general solution is

$$y(z) = c_1 J_0(z) + c_2 Y_0(z)$$

where $J_0(z)$ is the Bessel function of the first kind of order zero and $Y_0(z)$ is the Bessel function of the second kind of order zero.

The function $Y_0(z)$ is singular at $z = 0$ whereas $J_0(z)$ is an entire function. Obtain

$$R(\rho) = c J_0(k\rho)$$

where the boundary condition $R(a) = 0$ must still be enforced.

The Bessel function $J_0(z)$ has a sequence of positive zeros, $x_{0n}, n = 1, 2, \ldots$:

$$0 < x_{01} < x_{02} < x_{03} < \ldots$$

The boundary condition $R(a) = 0$ requires

$$0 = R(a) = c J_0(ka) \ .$$

Obtain:

$$k = k_n = \frac{1}{a} x_{0n}, \quad n = 1, 2, \ldots$$

Thus we obtain the following solutions of the wave equation satisfying the boundary condition $u = 0$ for $\rho = a$:

$$u_n(\rho, t) = \Big(A_n \cos(c k_n t) + B_n \sin(c k_n t) \Big) J_0(k_n \rho), \quad k_n = x_{0n}/a \ ,$$

for $n = 1, 2, \ldots$ Here A_n and B_n are free constants.

By superposition, any function

$$u(\rho, t) = \sum_{n=1}^{\infty} \Big(A_n \cos(ck_n t) + B_n \sin(ck_n t) \Big) J_0(k_n \rho), \quad k_n = x_{0n}/a \ ,$$

also solves the wave equation with the same boundary condition as long as the series converges to a smooth function and we can exchange differentiations with summation.

We now try to determine A_n and B_n so that the initial condition is satisfied. This requires

$$f(\rho) = u(\rho, 0) = \sum_{n=1}^{\infty} A_n J_0(k_n \rho)$$

and

$$g(\rho) = u_t(\rho, 0) = \sum_{n=1}^{\infty} B_n c k_n J_0(k_n \rho)$$

where

$$k_n = x_{0n}/a \ .$$

Thus, we need expansion theorems and orthogonality relations for the functions

$$\rho \to J_0(k_n \rho), \quad n = 1, 2, \ldots$$

One can prove the following orthogonality relations:

$$\int_0^a J_0(k_j \rho) J_0(k_n \rho) \rho \, d\rho = \delta_{jn} \frac{a^2}{2} J_1^2(x_{0j}) \ .$$

Here $J_1(z)$ is the Bessel function of the first kind of order one.

Let us assume the above orthogonality relations. Assume that

$$f(\rho) = u(\rho, 0) = \sum_{n=1}^{\infty} A_n J_0(k_n \rho) \ .$$

Then multiply by $J_0(k_j \rho)\rho$ and integrate over $0 \le \rho \le a$ to obtain

$$A_j = \frac{2}{a^2 J_1^2(x_{0j})} \int_0^a J_0(k_j \rho) f(\rho) \rho \, d\rho \ .$$

Similarly,

$$B_j = \frac{2}{a^2 c k_j J_1^2(x_{0j})} \int_0^a J_0(k_j \rho) g(\rho) \rho \, d\rho \ .$$

7.2 The 2D Wave Equation in a Disk

We now drop the assumption that f and g depend only on ρ. The initial condition reads

$$u(\rho, \phi, 0) = f(\rho, \phi), \quad u_t(\rho, \phi, 0) = g(\rho, \phi) \ .$$

Let

$$u(\rho, \phi, t) = \alpha(t) R(\rho) \Phi(\phi) \ .$$

Obtain

$$R\Phi\alpha'' = c^2 \Big(R''\Phi\alpha + \frac{1}{\rho} R'\Phi\alpha + \frac{1}{\rho} R\Phi''\alpha \Big) \ ,$$

thus

$$\frac{\alpha''(t)}{c^2\alpha(t)} = \frac{R''(\rho)}{R(\rho)} + \frac{1}{\rho}\frac{R'(\rho)}{R(\rho)} + \frac{1}{\rho^2}\frac{\Phi''(\phi)}{\Phi(\phi)} =: -k^2 \ .$$

As before,

$$\alpha''(t) + c^2 k^2 \alpha(t) = 0 \ .$$

Also,

$$\frac{R''(\rho)}{R(\rho)} + \frac{1}{\rho}\frac{R'(\rho)}{R(\rho)} + \frac{1}{\rho^2}\frac{\Phi''(\phi)}{\Phi(\phi)} + k^2 = 0 \ .$$

It follows that

$$\Phi''(\phi) + m^2\Phi(\phi) = 0, \quad m = 0, 1, \ldots$$

and

$$\rho^2 R''(\rho) + \rho R'(\rho) + (\rho^2 k^2 - m^2) R(\rho) = 0 \ .$$

Let

$$R(\rho) = y(k\rho) \ .$$

Obtain:

$$z^2 y''(z) + z y'(z) + (z^2 - m^2) y(z) = 0 \ ,$$

which is Bessel's equation of order m. The general solution is

$$y(z) = c_1 J_m(z) + c_2 Y_m(z)$$

where $J_m(z)$ is the Bessel function of the first kind of order m and $Y_m(z)$ is the Bessel function of the second kind of order m.

The function $Y_m(z)$ is singular at $z = 0$ whereas $J_m(z)$ is an entire function. Obtain

$$R_m(\rho) = cJ_m(k\rho) \ .$$

Each function $J_m(z)$ has a sequence of positive zeros, x_{mn}:

$$0 < x_{m1} < x_{m2} < x_{m3} < \dots$$

The boundary condition

$$0 = R_m(a) = cJ_m(ka)$$

yields

$$k_{mn} = x_{mn}/a, \quad m = 0, 1, \dots \quad \text{and} \quad n = 1, 2, \dots$$

One obtains the following solutions of the wave equation satisfying the boundary condition $u = 0$ for $\rho = a$:

$$u_{mn}(\rho, \phi, t) = \cos(ck_{mn}t) \Big(a_{mn} \cos(m\phi) + b_{mn} \sin(m\phi) \Big) J_m(k_{mn}\rho)$$

and

$$u^*_{mn}(\rho, \phi, t) = \sin(ck_{mn}t) \Big(a^*_{mn} \cos(m\phi) + b^*_{mn} \sin(m\phi) \Big) J_m(k_{mn}\rho)$$

We try to determine the coefficients a_{mn}, b_{mn} form the initial condition

$$f(\rho, \phi) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Big(a_{mn} \cos(m\phi) + b_{mn} \sin(m\phi) \Big) J_m(k_{mn}\rho) \ .$$

(Here, for $m = 0$, the coefficients b_{0n} are irrelevant.) The above expansion of $f(\rho, \phi)$ is a Fourier–Bessel expansion.

For fixed ρ we make a Fourier expansion of the function

$$\phi \to f(\rho, \phi) \ .$$

It has the form

$$f(\rho, \phi) = A_0(\rho) + \sum_{m=1}^{\infty} A_m(\rho) \cos(m\phi) + B_m(\rho) \sin(m\phi) \ .$$

Then we make a Bessel expansion of $A_m(\rho)$ and $B_m(\rho)$ in terms of the functions

$$\rho \to J_m(k_{mn}\rho), \quad n = 1, 2, \dots$$

7.3 Auxiliary Results on the Γ Function

Recall that

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt, \quad s > 0 \ .$$

Most important is the functional equation

$$\Gamma(s+1) = s\Gamma(s), \quad s > 0 \ .$$

For $s = \frac{1}{2}$ one obtains, using the substitution

$$t^{1/2} = \tau, \quad 2d\tau = t^{-1/2} dt \ ,$$

$$\begin{aligned} \Gamma(\frac{1}{2}) &= \int_0^\infty t^{-1/2} e^{-t}\, dt \\ &= 2\int_0^\infty e^{-\tau^2}\, d\tau \\ &= \sqrt{\pi} \end{aligned}$$

Let $k \in \{0, 1, 2, \ldots\}$. We have

$$\begin{aligned} \Gamma(\frac{1}{2}) &= \sqrt{\pi} \\ \Gamma(1+\frac{1}{2}) &= \frac{1}{2}\sqrt{\pi} \\ \Gamma(2+\frac{1}{2}) &= \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \\ \Gamma(3+\frac{1}{2}) &= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \\ \Gamma(k+1+\frac{1}{2}) &= \frac{1\cdot 3\ldots(2k+1)}{2^{k+1}}\sqrt{\pi} \end{aligned}$$

Here

$$1\cdot 3\cdot 5\ldots(2k+1) = \frac{(2k+1)!}{2^k k!} \ .$$

This yields

$$\Gamma\Big(k+1+\frac{1}{2}\Big) = \frac{(2k+1)!}{2^{2k+1}\, k!}\sqrt{\pi} \ .$$

7.4 Series Representation of $J_m(z)$

Let $m \geq 0$. We have seen that

$$J_m(x) = \Big(\frac{x}{2}\Big)^m \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+1+m)} \Big(\frac{x}{2}\Big)^{2k} \ .$$

We try to understand the behavior of $J_m(x)$ for $x \geq 0$,

First, let $0 \leq x < \varepsilon$. We have

$$J_m(x) = \left(\frac{x}{2}\right)^m \left(\frac{1}{\Gamma(1+m)} + \mathcal{O}(x^2)\right)$$

with

$$\Gamma(1+m) > 0 \ .$$

Clearly,

$$J_0(0) = 1 \ .$$

If $m > 0$ then $J_m(x)$ vanishes to order m at $x = 0$ and is positive for $0 < x < \varepsilon$.

We can use this series representation to show the following:

Lemma 7.1 *For $m = \frac{1}{2}$ we have*

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad x > 0 \ .$$

Proof: We have

$$\begin{aligned}
J_{1/2}(x) &= \left(\frac{x}{2}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1+\frac{1}{2})} \left(\frac{x}{2}\right)^{2k} \\
&= \left(\frac{2}{\pi x}\right)^{1/2} \cdot \frac{x}{2} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k\, 2^{2k+1}}{(2k+1)!} \left(\frac{x}{2}\right)^{2k} \\
&= \left(\frac{2}{\pi x}\right)^{1/2} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{(2k+1)!} \\
&= \left(\frac{2}{\pi x}\right)^{1/2} \sin x
\end{aligned}$$

It follows that the zeros of $J_{1/2}(x)$ are

$$x_{1/2,n} = n\pi, \quad n = 0, 1, 2 \ldots$$

7.5 The zeros of $J_m(x)$

Let $m \geq 0$. We show here how to use Sturm's theorem to obtain information about the zeros of $J_m(x)$.

Recall that $J_m(x)$ satisfies Bessel's equation:

$$x^2 J_m''(x) + x J_m'(x) + (x^2 - m^2) J_m(x) = 0 \ .$$

Fix m and define the function $w_m(x)$ by

$$J_m(x) = x^{-1/2}\, w_m(x), \quad x > 0 \ .$$

(This is the Liouville transform to obtain an equation for $w_m(x)$ without first derivative term. In fact, $p(x) = 1/x$ and $P(x) = \ln x$ and $e^{-P(x)/2} = x^{-1/2}$.)

Lemma 7.2 *The function $w_m(x)$ satisfies the differential equation*

$$w_m''(x) + \Big(1 - \frac{m^2 - \frac{1}{4}}{x^2}\Big) w_m(x) = 0, \quad x > 0 \ .$$

Prrof: This is a simple computation.

Remark: For $m = 1/2$ the above differential equation has constant coefficients, $w_{1/2}'' + w_{1/2} = 0$. This is consistent with the above result for $J_{1/2}(x) = cx^{-1/2} \sin x$.

Case 1: Let $0 \le m < \frac{1}{2}$. We apply Sturm's theorem with

$$g_1(x) \equiv 1, \quad g_2(x) = 1 - \frac{m^2 - \frac{1}{4}}{x^2} \ .$$

It is clear that $g_2(x) > 1$ for all $x > 0$. Let

$$y_1(x) = \sin(x - \alpha)$$

and

$$y_2(x) = w_m(x) \ .$$

Here $\alpha \ge 0$ is arbitrary. The function $y_1(x)$ has zeros

$$p = \alpha < q = \alpha + \pi \ .$$

By Sturm's theorem, the function $y_2 = w_m$ has a zero between α and $\alpha + \pi$. Since α is arbitrary, we obtain that the function $J_m(x)$ has a sequence of positive zeros, denoted by $x_{mn}, n = 1, 2, \ldots$:

$$0 < x_{m1} < x_{m2} < x_{m3} < \ldots$$

(If $m > 0$ then $x_{m0} = 0$ is also a zero of $J_m(x)$.)

For fixed m, the zeros of $J_m(x)$ cannot accumulate at some finite value $\bar{x}$. Otherwise, one would obtain that

$$J_m(\bar{x}) = J_m'(\bar{x}) = 0 \ ,$$

and $J_m(x) \equiv 0$ would be implied.

Furthermore, we claim that, for $0 \le m < \frac{1}{2}$:

$$x_{m,n+1} - x_{m,n} < \pi \ .$$

In other words, any two consecutive zeros of $J_m(x)$ have a distance less than π for $0 \le m < \frac{1}{2}$. This follows from Sturm's theorem applied with

$$y_1(x) = \sin(x - x_{m,n}) \ .$$

We summarize:

Theorem 7.1 *Let $0 \le m < \frac{1}{2}$. The function $J_m(x)$ has infinitely many positive zeros. These can be ordered as a sequence:*

$$0 < x_{m1} < x_{m2} < x_{m3} < \ldots$$

We have

$$x_{mn} \to \infty \quad as \quad n \to \infty$$

and

$$x_{m,n+1} - x_{m,n} < \pi, \quad n = 1, 2, \ldots$$

Case 2: $m > \frac{1}{2}$. The function $w_m(x)$ satisfies

$$w_m''(x) + g_2(x) w_m(x) = 0$$

with

$$g_2(x) = 1 - x^{-2}\Big(m^2 - \frac{1}{4}\Big) < 1 \ .$$

For $x > m$ we have

$$g_2(x) > g_2(m) = \frac{1}{4m^2} =: g_1(x) \ .$$

We know that

$$y_1(x) = \sin\Big(\frac{x}{2m} - \alpha\Big)$$

solves

$$y_1'' + \frac{1}{4m^2} y_1 = 0$$

and y_1 has infinitely many positive zeros. It follows that, again, $J_m(x)$ has a sequence of positive zeros,

$$0 < x_{m1} < x_{m2} < x_{m3} < \ldots$$

We claim that

$$x_{m,n+1} - x_{m,n} > \pi \ .$$

This follows from

$$g_2(x) < 1$$

since the solutions $y_3(x)$ of

$$y_3'' + y_3 = 0$$

have zeros with distance π. We consider

$$y_3(x) = \sin(x - x_{m,n}) \ .$$

Then, by Sturm's theorem, y_3 has a zero strictly between $x_{m,n}$ and $x_{m,n+1}$, which yields $x_{m,n+1} - x_{m,n} > \pi$.

We summarize:

Theorem 7.2 *Let $m > \frac{1}{2}$. The function $J_m(x)$ has infinitely many positive zeros. These can be ordered as a sequence:*

$$0 < x_{m1} < x_{m2} < x_{m3} < \ldots$$

We have

$$x_{mn} \to \infty \quad as \quad n \to \infty$$

and

$$x_{m,n+1} - x_{m,n} > \pi, \quad n = 1, 2, \ldots$$

7.6 Remarks on Matlab

The commands

```
x=0:0.01:20
y=besselj(1,x)
plot(x,y)
```

give a plot of $J_1(x)$ for $0 \leq x \leq 20$.

To find zeros, one can use

x=fzero(fun,x0)

after defining the function fun in a .m–file.