PARTIAL DIFFERENTIAL EQUATIONS IN PHYSICS

LECTURES ON THEORETICAL PHYSICS, VOL. VI

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CHAPTER V

Eigenfunctions and Eigen Values

In this chapter we shall develop Fourier's methods to their greatest generality and thereby open up the boundary value problems of physics to mathematical treatment. The most striking demonstration of the power of these methods was given in 1926 when Erwin Schrödinger recognized the quantum numbers as eigen values of his wave equation and thereby put the tools of modern analysis at the service of atomic physics. It was fortunate that he had the aid of his Zürich colleague Hermann Weyl who had, as the greatest pupil and later the successor of Hilbert in Göttingen, an essential part in the development of the theory of integral equations. However we should note that, while the viewpoint of *integral equations* is important for the rigorous mathematical foundation, in particular for the existence proofs for the eigenfunctions and their eigen values, the older viewpoint of *partial differential equations* leads to the same concepts in a natural manner. We shall start by demonstrating this with an example which was known long before integral equations.

§ 25. Eigen Values and Eigenfunctions of the Vibrating Membrane

The subject of the following consideration is a membrane without proper elasticity (see p. 33) which is clamped into a frame whose resistance to distortion is entirely due to the stresses working on its edge. We consider these stresses as perpendicular to the edge in the plane of the membrane. For the deformed membrane this results in a pressure N which acts perpendicular to the surface and is equal to T times the mean curvature of the membrane, and hence is equal to $T\,\Delta u$ for a small deformation u. The wave equation (7.4) for a pure harmonic oscillation of frequency ω then yields

$$-\sigma\,\omega^2\,u = T\,\Delta u, \qquad \sigma = \text{surface density.}$$

This we rewrite in the customary form

$$\Delta u + k^2\,u = 0, \qquad k^2 = \frac{\sigma\,\omega^2}{T}. \tag{1}$$

If we do not consider k^2 as constant but as an arbitrary function $F(x,y)$

then according to (10.6) this is the general linear *self-adjoint* elliptic differential equation of second order in two variables in its normal form.

The non-trivial solutions of (1) which satisfy the boundary condition $u = 0$ are called *eigenfunctions* and the corresponding k are called the *eigen values* of the problem. If k^2 or $F(x,y)$ were negative then no eigen values would exist, as we saw in the introduction to exercise II.2. The fact that eigen values do exist for positive k^2, namely, an *infinite number*, can be shown first for the simplest examples.

a) The rectangle $0 \leq x \leq a$, $0 \leq y \leq b$. The boundary conditions are satisfied by

$$u = u_{nm} = \sin n\pi\frac{x}{a}\sin m\pi\frac{y}{b}, \begin{cases} n = 1, 2, \cdots\infty, \\ m = 1, 2, \cdots\infty. \end{cases} \tag{2}$$

From the differential equation we then have

$$k = k_{nm} = \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}. \tag{2a}$$

We shall ignore the constant factor by which the solutions can be multiplied. We first assume a and b to be *incommensurable*. Then all the k_{nm} are different and only *one* eigenfunction u corresponds to each k. The number of eigen values is *infinite*.

b) Circle, circular ring, circular sector. For the *full circle* $0 \leq r \leq a$ we can write:

$$u = I_m(k\,r)\,e^{\pm i m\varphi}, \quad m = 0, 1, 2, \ldots\infty, \tag{3}$$

where k satisfies the boundary condition

$$I_m(k\,a) = 0. \tag{3a}$$

Since this equation has infinitely many roots (see Fig. 21) there are again *infinitely many eigen values* $k = k_{nm}$. The roots of (3a) are all different, but for $m > 0$ there are two eigenfunctions for each eigen value corresponding to the different signs in (3), or, in other words, corresponding to the double possibility $\begin{matrix}\cos\\ \sin\end{matrix} m\varphi$. We say that the problem is *degenerate* for $m > 0$; in our case it is *simply degenerate*. According to (20.4b) the (non-degenerate) basic tone of the circular membrane is $k_{10} = 2.40/a$.

For the *circular ring* $b \leq r \leq a$ we write

$$u = [I_m(k\,r) + c\,N_m(k\,r)]\,e^{\pm i m\varphi}. \tag{4}$$

Here we need both particular solutions I and N of the Bessel differential

equation (we could, of course, consider H^1 and H^2 instead) in order to be able to satisfy the two boundary conditions:

$$I_m(k\,a) + c\,N_m(k\,a) = 0,$$
$$I_m(k\,b) + c\,N_m(k\,b) = 0. \tag{4a}$$

Here too there exists an infinite number of different k_{nm} with their associated c_{nm}. This problem, too, is simply degenerate when $m > 0$, since, according to (4), there are then two different u_{nm} for each k_{nm}.

For the *circular sector* $0 \leq r \leq a$, $0 \leq \varphi \leq \alpha$ we set

$$u = I_\mu(k\,r)\,\sin\,\mu\varphi, \qquad \mu = m\,\frac{\pi}{\alpha}, \tag{5}$$

where the k are determined by the condition $I_\mu(ka) = 0$. Infinitely many eigen values $k = k_{nm}$ exist; the problem is not degenerate.

The most general region which can be treated in this manner is the *circular ring sector* $b \leq r \leq a$, $0 \leq \varphi \leq \alpha$, which is bounded by two circular arcs and two radii.

c) Ellipse and elliptic-hyperbolic curvilinear quadrangle. The wave equation (1) written in elliptic coordinates ξ, η can be separated (see v.II exercise IV.3) and leads to a so-called Mathieu equation in each coordinate. The solution $\xi = \text{const}$ yields the ellipses which belong to the family of curves; $\eta = \text{const}$ yields the hyperbolas of the family. For the full ellipse we have, in addition to the boundary condition $u = 0$, a condition of continuity for $\xi = 0$ (focal line) and the condition of periodicity for $\eta = \pm\pi$. The determination of the eigen values leads to complicated transcendental equations which we cannot discuss here. The most general region of this kind is the curvilinear quadrangle whose boundary consists of two elliptic arcs and two arcs of hyperbolas which are confocal with the former.

The simple examples which we considered here are special cases of the *fundamental theorem* of the theory of oscillating systems with infinitely many degrees of freedom and their eigenfunctions: *For an arbitrary region an infinite sequence of eigen values k exists for which there is a solution of the corresponding differential equation $\Delta u + k^2 u = 0$ which is continuous in the interior of the region and satisfies the boundary condition $u = 0$* (or any of the other boundary conditions on p. 63).[1] The problem of finding a rigorous proof for this theorem has repeatedly challenged the ingenuity of mathematicians, starting with Poincaré's great work (*Rendic. Circ. Math. di Palermo*, 1894) and culminating in the

[1] The same theorem holds for the eigen value λ of the general self-adjoint differential equation $\Delta u + \lambda F(x, y)\,u = 0$, $F > 0$.

Fredholm-Hilbert theory of integral equations. Here we must be satisfied with proving the related theorem for mechanical systems with a finite number of degrees of freedom: *A system with f degrees of freedom which is in stable equilibrium, can have exactly f linearly independent small (or more precisely, infinitely small) sine-like oscillations about this state.*

We write the kinetic energy for the neighborhood of a state of equilibrium $q_1 = q_2 = \ldots = q_f = 0$ in the form:

$$T = \frac{1}{2}\sum\sum a_{nm}\,\dot{q}_n\,\dot{q}_m.$$

Because q is so small we consider the a_{nm} as constants. At the same time the potential energy V becomes a quadratic form in the q_n with constant coefficients since the linear terms $\partial V/\partial q_n$ vanish in the expansion of V in terms of the q_i around the state of equilibrium

$$V - V_0 = \frac{1}{2}\sum\sum b_{nm}\,q_n\,q_m.$$

Now it is always possible to transform *both* the above quadratic forms simultaneously into sums of squares by a linear transformation (transformation to principal axes of quadratic surfaces). Performing this transformation we obtain:

$$T = \frac{1}{2}\sum a_n\,\dot{x}_n^2, \qquad V - V_0 = \frac{1}{2}\sum b_n\,x_n^2.$$

The new coordinates x_n are called *normal coordinates* of the system. According to the Lagrange equation we then have

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_n} = -\frac{\partial V}{\partial x_n}, \quad \text{hence} \quad a_n\,\ddot{x}_n = -b_n\,x_n.$$

T is a positive definite quadratic form and so is $V - V_0$ for a stable equilibrium; hence the a_n and b_n are positive. Thus for every normal coordinate we obtain a stable oscillation

$$x_n = c_n\,e^{i\omega_n t} \quad \text{with} \quad \omega_n^2 = \frac{b_n}{a_n} > 0,$$

which gives as many oscillations as there are degrees of freedom. In the limit $f \to \infty$ there corresponds an eigen value k_n to every ω_n, and to the totality of $q_1, \ldots, q_n$ that belong to the individual x_n there now corresponds the eigenfunction u_f. The k and the ω_n are both *real*.

We point out that the fact that the k are real can also be proved directly from the differential equation without passing to the limit. If a k were complex then the corresponding u would be complex and the conjugate function u^* would have to satisfy the conjugate differential equa-

tion $\Delta u^* + k^{*2} u^* = 0$ with the boundary condition $u^* = 0$. From Green's theorem

$$\int (u\,\Delta u^* - u^*\,\Delta u)\,d\sigma = \int \left(u\frac{\partial u^*}{\partial n} - u^*\frac{\partial u}{\partial n}\right) ds, \tag{6}$$

where the right side vanishes due to the boundary conditions; it follows that

$$(k^2 - k^{*2}) \int u\,u^*\,d\sigma = 0.$$

But uu^* is always $\geqq 0$; hence the integral cannot vanish and hence we must have $k = k^*$, and k must be real. The physical meaning of the real character of the k is that under our conditions the oscillation process is always free from absorption.

Up to now we have assumed our problem to be *non-degenerate*. However, for the perturbation theory of wave mechanics the degenerate cases are of special interest. We return to our example of the rectangle and no longer assume the sides a,b, to be incommensurable. This is certainly the case for the *square* $a = b$. Then we obtain from (2a)

$$k_{nm} = \frac{\pi}{a}\sqrt{n^2 + m^2}, \quad \text{hence} \quad k_{nm} = k_{mn};$$

but according to (2) we have $u_{nm} \neq u_{mn}$, unless $n = m$, namely,

$$u_{n\,m} = \sin n\pi\frac{x}{a}\,\sin m\pi\frac{y}{a},$$

but

$$u_{m\,n} = \sin m\pi\frac{x}{a}\,\sin n\pi\frac{y}{a},$$

All oscillations with $n \neq m$ are therefore (at least) *simply degenerate*, since two different types of oscillations u_{nm} and u_{mn} correspond to the same k_{mn}. Only the basic oscillation k_{11} and its overtones $k_{nn} = n\,k_{11}$ (which in this special case are harmonic) are *non-degenerate*.

Let us examine somewhat more closely the cases $n = 1$, $m = 2$ and $n = 2, m = 1$ (hence $k_{12} = k_{21} = \sqrt{5}\,\pi/a$). In Figs. 23 and 24 we characterize the corresponding eigenfunctions by their nodal lines. These are the lines $u = 0$ in which powder strewn on the membrane would collect. Together with u_{12} and u_{21} we have, belonging to $k_{12} = k_{21}$, the eigenfunctions

$$u = u_{12} + \lambda\,u_{21}, \tag{7}$$

where λ is an arbitrary constant. By a continuous

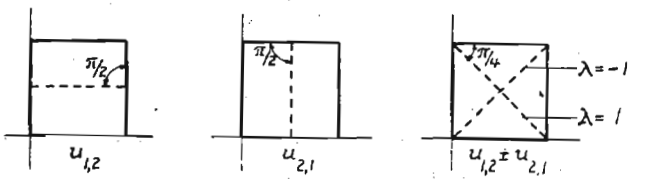

Fig. 23. Simple degeneration in the case of a quadratic membrane for $n = 1$, $m = 2$ or $m = 1$, $n = 2$. The diagonals are the nodal lines for $\lambda = \pm 1$.

deformation of λ the form of the nodal lines within the family (7) is continuously deformed. We compute the linear combinations with $\lambda = \pm 1$:

$$u = \sin\pi\frac{x}{a}\sin 2\pi\frac{y}{a} \pm \sin 2\pi\frac{x}{a}\sin\pi\frac{y}{a}$$

$$= 2\sin\pi\frac{x}{a}\sin\pi\frac{y}{a}\left(\cos\pi\frac{y}{a} \pm \cos\pi\frac{x}{a}\right).$$

From the last expression here we see that the diagonal $y = x$ is a nodal line of $\lambda = -1$ while the other diagonal $y = a - x$ belongs to $\lambda = +1$. Fig. 24 shows the behavior of the lines for arbitrary values of the parameter λ.

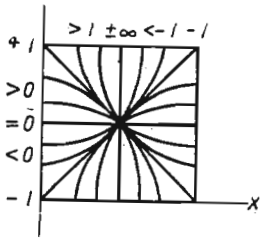

Fig. 24. Total picture of the possible nodal lines for the quadratic membrane. The numbers on the left and on top are the values of the parameter λ in (7).

Under certain conditions higher degenerations occur in the case of the quadratic membrane. For example, if we have

$$n_1^2 + m_1^2 = n_2^2 + m_2^2;$$

then for the eigen value

$$k = \frac{\pi}{a}\sqrt{n_1^2 + m_1^2} = \frac{\pi}{a}\sqrt{n_2^2 + m_2^2}$$

we have four linearly independent eigenfunctions

$$u_{n_1\,m_1},\quad u_{n_2\,m_2},\quad u_{m_1\,n_1},\quad u_{m_2\,n_2}.$$

Hence we have a case of triple degeneration. The higher degeneration here depends on whether or not a number can be expressed as the sum of two squares in more than one way, as for example

$$65 = 1^2 + 8^2 = 4^2 + 7^2.$$

According to Gauss' *Disquisitiones Arithmeticae* this is the case for every sum of two squares among whose prime factors there are at least two

different ones of the form $4n + 1$. Such primes permit the complex decomposition

$$4n + 1 = (a + bi)(a - bi)$$

with integral a,b; and the different groupings of the complex factors lead to different representations as a sum of squares. In our example $65 = 5.13$ we have

$$5 = (1 + 2i)(1 - 2i) \text{ and } 13 = (2 + 3i)(2 - 3i)$$

and hence

$$65 = \begin{cases} (1+2i)\ (2+3i)\cdot(1-2i)\ (2-3i) = (-4+7i)\ (-4-7i) \\ \qquad\qquad = 4^2 + 7^2, \\ (1+2i)\ (2-3i)\cdot(1-2i)\ (2+3i) = (8+i)\cdot(8-i) \\ \qquad\qquad = 8^2 + 1^2. \end{cases}$$

For any two eigenfunctions u,u' with $k \neq k'$ we have the *orthogonality theorem*

$$\int u\, u'\, d\sigma = 0 \tag{8}$$

as a result of Green's theorem. The proof is the same as in (6) if we replace u^* by u'. But this deduction fails if u and u' belong to the same degenerate state, in other words if $k = k'$.

In order to avoid cumbersome considerations of special cases, it is desirable to force orthogonality also in the degenerate cases. It will prove convenient to introduce the abbreviation of Courant-Hilbert[2] for the integral in (8):

$$\int u\, u'\, d\sigma = (u, u'). \tag{8a}$$

In §26 we shall return to a discussion of the connection between this expression and the scalar product in ordinary vector analysis. We call the integral in (8a) the "scalar product" of u and u'.

We first prove the theorem that n continuous, real, mutually orthogonal but otherwise arbitrary functions $u_1, u_2, \ldots, u_\nu, \ldots, u_n$ are *linearly independent*. For if there existed an equation of the form

$$\sum_{\nu=0}^{n} c_\nu u_\nu = 0 \quad \text{with} \quad (u_\mu, u_\nu) = 0 \ \text{ for all } \ \mu \neq \nu\ ,$$

then by the "scalar multiplication" of the equation by u_μ we would obtain

[2] Courant-Hilbert, Methoden der mathematischen Physik, 2nd ed., Springer, Berlin 1931, Chapter II.

$$c_\mu(u_\mu, u_\mu) = 0, \text{ hence } \quad c_\mu = 0 \quad \text{for all } \ \mu,$$

which contradicts the assumption of linear dependence.

We now proceed step by step and first treat the case of *simple degeneracy*. Let u_1,u_2 be continuous, real, not necessarily orthogonal functions belonging to the same eigen value. We consider the family

$$u = c_1u_1 + c_2u_2$$

and consider the member which is orthogonal to u_1. This member is given by the condition

$$0 = (u_1, u) = c_1 (u_1, u_1) + c_2 (u_1, u_2).$$

We satisfy this condition by setting

$$c_1 = -(u_1, u_2),\ c_2 = (u_1, u_1) \tag{9}$$

where $c_2 \neq 0$ and hence $u \neq 0$. In u_1 and u we have two mutually orthogonal eigenfunctions of the family, which we choose as the representatives of the family instead of u_1,u_2. We now can normalize u by multiplication with a constant factor such that

$$(u,u) = (u_1,u_1). \tag{9a}$$

For *twofold degeneracy* let u_1,u_2 be two functions that are normalized according to (9) and (9a), and let u_3 be a function of the same eigen value that is not necessarily orthogonal to the first two. We consider the family

$$u = c_1u_1 + c_2u_2 + c_3u_3$$

and select the member of the family that is orthogonal to both u_1 and u_2, thus obtaining the conditions

$$0 = (u_1, u) = c_1 (u_1, u_1) + c_3 (u_1, u_3),$$
$$0 = (u_2, u) = c_2 (u_2, u_2) + c_3 (u_2, u_3).$$

We satisfy both conditions by setting

$$c_1 = -(u_1, u_3),\ c_2 = -(u_2, u_3),\ c_3 = (u_1, u_1) = (u_2, u_2). \tag{10}$$

The functions u_1,u_2, u are mutually orthogonal and hence linearly independent; furthermore we can normalize u so as to obtain:

$$(u,u) = (u_1,u_1) = (u_2,u_2). \tag{10a}$$

We thereby obtain the desired orthogonalization for twofold degeneracy.

This process can obviously be continued in the case of higher

degeneracy. The degenerate eigenfunctions are thus made mutually orthogonal; due to (8) they are already orthogonal to the eigenfunctions which belong to different k.

To the orthogonality condition (8) we add the *normality condition*

$$(u,\ u) = \int u^2\, d\sigma = 1 \tag{11}$$

This "normalization to 1" leads to a certain simplification of the orthogonalization process above (see, e.g. (10a)). We shall see in §26 that (11) also has its vector-analytic analog. We still must mention that for complex u equation (11) must be amended to read

$$(u\, u^*) = \int u\, u^*\, d\sigma = 1 \tag{11a}$$

and that in separable problems the normalization is best carried out for each individual factor. Thus in (2) we have to multiply the sine functions by the factors

$$\sqrt{\frac{2}{a}} \quad \text{and} \quad \sqrt{\frac{2}{b}} \quad \text{respectively} \tag{12}$$

and in (3) we have to multiply the exponential function and the Bessel function by the factors

$$\frac{1}{\sqrt{2\pi}} \quad \text{and} \quad \frac{\sqrt{2}}{a\, I'_m(ka)} \quad \text{respectively} \tag{12 a}$$

(the latter is due to (20.9a)). Thus our solutions in (2) and (4) at the beginning of this section are determined also with respect to their amplitudes.

From the above-mentioned examples we deduce two theorems concerning nodal lines, which we shall prove now for membranes with arbitrary boundaries:

1. If several nodal lines intersect at a point then they intersect at equal angles (isogonally): for two such lines the angle is $\pi/2$, for ν lines it is π/ν.

2. The larger the eigen value k, the finer the subdivision of the membrane into regions of alternating signs; for $k \to \infty$ the nodal lines become everywhere dense.

In order to demonstrate that theorem 1 holds for our special examples we refer to Figs. 23 and 24, where the boundary itself must be considered a nodal line and the angles are $\pi/2$ and $\pi/4$ as shown. In the case of the full circle we see from (3) that there are m radial lines intersecting at its center at an angle of π/m. In order to show that theorem 2 is satisfied in our cases it suffices to note that for the case of the rectangle and the eigen value k_{nm} the rectangle is subdivided into sub-

rectangles of sides a/n and b/m, so that for $k \to \infty$ at least one side approaches zero.

For the proof of theorem 1 we develop u in the neighborhood of the point O in a Fourier series. We use an r, φ - coordinate system whose origin is at O. For any shape of the membrane we obtain the expansion

$$u = \sum_n I_n(kr)\,(a_n \cos n\varphi + b_n \sin n\varphi) \tag{13}$$

which converges in a certain neighborhood of O, where the a,b are determined coefficients which can be computed from the given u. The fact that the radial functions in the Fourier expansion must be the Bessel functions I_n follows from differential equation (1) and the regularity of u at O. Now if there is to exist *at least* one nodal line through the point O ($r = 0$), then according to (13)

$$0 = I_0(0)\, a_0, \quad \text{and hence } a_0 = 0.$$

Then if a_1 and b_1 are not both zero there is *only* one line through O whose direction is determined by the equation:

$$0 = I_1(kr)\ (a_1 \cos\varphi + b_1 \sin\varphi).$$

Hence for $r > 0$

$$\tan\varphi = -\frac{a_1}{b_1}.$$

This determines the direction of the nodal line *uniquely*.

If there is to be more than one nodal line through O then we must have $a_1 = b_1 = 0$. If we do not at the same time have $a_2 = b_2 = 0$ then according to (13) we have

$$0 = I_2(kr)\,(a_2 \cos 2\varphi + b_2 \sin 2\varphi)$$

or, if there are to be ν nodal lines through O, and hence all a,b up to but not including a_ν, b_ν vanish, then we have

$$0 = I_\nu(kr)\,(a_\nu \cos\nu\varphi + b_\nu \sin\nu\varphi).$$

In the latter case we have for $r > 0$

$$\tan\nu\varphi = -\frac{a_\nu}{b_\nu}. \tag{13a}$$

The right side of this equation is given by our Fourier expansion and shall be denoted by $\tan\alpha$. The general solution of (13a) is then:

$$\varphi = \alpha, \quad \alpha + \frac{\pi}{\nu}, \quad \alpha + \frac{2\pi}{\nu}, \ \ldots, \ \alpha + \frac{(\nu-1)\pi}{\nu}. \tag{13b}$$

These angles differ by the constant amount π/ν, which proves the isogonality.

Passing to the proof of theorem 2, we consider two functions u,v where u is a solution of (1) that satisfies the given boundary condition and v is the special solution

$$v = I_0(kr).$$

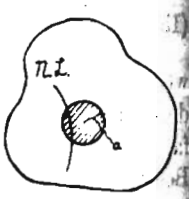

Fig. 25. With increasing k the nodal lines become denser and denser regardless of the shape of the membrane. The proof is given by considering a small disc anywhere on the membrane whose radius a decreases to zero for increasing k. N.L. stands for a nodal line which intersects the disc.

The value of k which is common to u and v is assumed to be large. With the help of this large k we define a small length a by setting $k\,a = \varrho_1$ where ϱ_1 is the first root of the equation $I_0(\varrho) = 0$. We consider a circular disc of radius a situated anywhere on the nodal line pattern of the eigenfunction u (see Fig. 25). With this disc as our domain of integration we apply Green's theorem:

$$\int (u\,\Delta v - v\,\Delta u)\,d\sigma = \int \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds. \tag{14}$$

The left side vanishes since both u and v satisfy the differential equation (1) with the same k. On the right side we have for $r = a$

$$v = 0 \text{ and } \frac{\partial v}{\partial n} = k\,I_0'(\varrho_1) \neq 0.$$

If we set $ds = a\,d\varphi$ then equation (14) becomes

$$\varrho_1 I_0'(\varrho_1) \int_0^{2\pi} u\,d\varphi = 0,$$

and hence

$$\int_0^{2\pi} u\,d\varphi = 0. \tag{15}$$

According to this u assumes both positive and negative values on the circumference of the disc. Hence there must be at least two zeros of u on the circumference; that is, our disc must be intersected by at least one nodal line. The disc becomes smaller as k becomes larger and hence for increasing eigen value k the nodal lines become *arbitrarily dense*. This holds for every part of the nodal line pattern.

§ 26. General Remarks Concerning the Boundary Value Problems of Acoustics and of Heat Conduction

The eigenfunctions of the oscillating membrane can be adapted directly to the spatial case. Here we do not think of an oscillating rigid body, but (in order to avoid all complications involving vectors and tensors) rather of an oscillating air mass in the interior of a closed rigid hull of finite extension. Just as on p.166, we interpret the scalar function u as the velocity potential of the air oscillations and we again set the boundary condition $\partial u/\partial n = 0$.

For the rectangular solid with side lengths a,b,c we have, in analogy to (25.2),

$$u = u_{nml} = \cos n\pi\frac{x}{a}\cos m\pi\frac{y}{b}\cos l\pi\frac{z}{c} \tag{1}$$

with eigen value

$$k = k_{nml} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{c^2}}. \tag{1a}$$

This state is non-degenerate if a,b,c are incommensurable.

For a sphere of radius a we obtain the general eigenfunction in analogy to (25.3):

$$u = u_{nml} = \psi_n(k_{nl}r)\,P_n^m(\cos\vartheta)\,e^{im\varphi} \tag{2}$$

Under our boundary condition the eigen value is given by

$$\psi_n'(k_{nl}a) = 0, \tag{2a}$$

where k_n is the l-th root of this equation. This state is $2n$-fold degenerate, since k_{nl} is independent of m and the different states P_n^m for upper index $-n \leq m \leq +n$ belong to the same k_{nl}.

Also in this category are the eigenfunctions of the circular cylinder $(0 < r < a,\ 0 < z < h)$ which we derived in §20 C. With the boundary condition $\partial u/\partial n = 0$ they are given by

$$u_{nlm} = I_n(\lambda r)\,e^{\pm in\varphi}\cos m\pi\frac{z}{h}; \tag{3}$$

the corresponding eigen value is determined from the equation $I_n'(\lambda a) = 0$, the l-th solution of which we denote by λ_{nl}. Therefore

$$k_{nlm}^2 = \lambda_{nl}^2 + m^2\pi^2/h^2. \tag{3a}$$

Due to the factor $\exp(\pm in\varphi)$ in (3) this state is simply degenerate for $n > 0$.

We now consider these eigenfunctions "normalized to 1" where we have to keep in mind the remarks on pp. 173,174. Then for example in (1) we have to replace $\cos\, n\pi\, x/a$ by

$$\cos n\pi \frac{x}{a} \sqrt{\frac{2}{a}}$$

and according to (22.31b) we have to replace P_n^m in (2) by

$$\Pi_n^m = P_n^m \quad \sqrt{(n+\tfrac{1}{2})\frac{(n-m)!}{(n+m)!}},$$

etc. (see exercise V.1).

We now generalize the fundamental theorem on p. 169 and its (mathematically non-rigorous) proof to the case of an arbitrary spatial region S. The theorem now reads: *There exists an infinite system of eigenfunctions*

$$u_1, u_2, \ldots, u_n, \ldots,$$

whose elements are regular in the interior of S and satisfy the differential equation

$$\Delta u_n + k_n^2 u_n = 0,$$

as well as a homogeneous boundary condition. The corresponding eigen values

$$k_1, k_2, \ldots, k_n, \ldots,$$

ordered in an increasing sequence, are infinite in number and increase to infinity; if S is bounded then they form a "discrete spectrum" and they are real since the differential equation was assumed free from absorption.

This system of eigenfunctions satisfies the conditions of *orthogonality* and of *normality:*

$$\int u_n u_m \, d\tau = \delta_{nm}, \tag{4}$$

which according to (25.8a) can be written as

$$(u_n, u_m) = \delta_{nm} \tag{4a}$$

or for complex eigenfunctions

$$(u_n, u_m^*) = \delta_{nm}. \tag{4b}$$

If the system of u_n is *complete* (see p. 5) then we claim that any continuous point function f given on S can be expanded in the u_n:

$$f = \sum A_n u_n. \tag{5}$$

If this expansion is possible then, according to (4b), we obtain from (5) through termwise integration

$$A_n = \int f u_n^* \, d\tau. \tag{5a}$$

That this expansion is possible is postulated by the *Ohm-Rayleigh principle*, which we shall assume in the following discussion without presenting its mathematical proof. In connection with the name of this principle we remark: Georg Simon Ohm was not only the discoverer of the basic law of Galvanic conduction, but also did profound research in acoustics. He found that the differences in the tone-color of different musical instruments are the result of differences in the mixture of basic tone and overtones. Since, according to (25.1), the overtones ω_n are related to the k_n, and since they are harmonic with the basic tone only for strings and organ pipes, so the construction of an arbitrary tone-color means the construction of an arbitrary function from the (in general anharmonic) eigen values. In Lord Rayleigh's classic book, *Theory of Sound*, this principle is generalized in the sense of equation (5) and is applied in many directions.

We shall now make some remarks about so-called *Hilbert space*, not only to justify the notation (u_n, u_m) of (4a,b) which is reminiscent of vector analysis, but also to give the Ohm-Rayleigh principle an elegant geometric interpretation, which in the hands of the Hilbert school has even been worked out as a means of proving this principle.

In accord with Courant-Hilbert (see p. 172) we define, in a space of N dimensions, the basis vectors $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_N$ (corresponding to the i,j,k of three-dimensional vector analysis) which lie in the coordinate directions $x_1, x_2, \ldots, x_N$ and whose scalar product is to satisfy the condition

$$(\mathfrak{e}_n \mathfrak{e}_m) = \delta_{nm} \tag{6}$$

We further consider a vector which forms the angles $\alpha_1, \alpha_2, \ldots$ with the coordinate axes

$$\mathfrak{a} = \cos\alpha_1\, \mathfrak{e}_1 + \cos\alpha_2\, \mathfrak{e}_2 + \cdots + \cos\alpha_N\, \mathfrak{e}_N \tag{7}$$

and we call it a *unit vector* if the scalar product of $\mathfrak{a}$ with itself has the value 1:

$$(\mathfrak{a},\mathfrak{a}) = \sum_{n=1}^{N} \cos^2\alpha_n = 1. \tag{7a}$$

A second unit vector $\mathfrak{b}$ with direction angles β_n is called *orthogonal* to

a if the scalar product of a and b vanishes:

$$(7b) \qquad (\mathfrak{a},\mathfrak{b}) = \sum_{n=1}^{N} \cos\alpha_n \cos\beta_n = 0.$$

Equations (7a,b) are seen to be generalizations of well-known formulas from three-dimensional analytic geometry.

In the limit $N \to \infty$ we now obtain Hilbert space. Here we note a formal analogy between the basis vectors $\mathfrak{e}_n$ and the elements u_n of our system of eigenfunctions. The relations between the latter as written in the form (4a) are formally the same as the relations (6) between the $\mathfrak{e}_n$. The system u_n, if it is complete, can serve as substitute for the basis $\mathfrak{e}_n$. The same is true for the u_n^* in the case of complex u_n. Every other system of functions that is orthogonalized and normalized to 1 can be composed from the u_n in the sense of equation (7) and can be visualized as a *vector* in Hilbert space. Two such vectors can be transformed into each other by a *rotation* of Hilbert space. But according to (5) any function f is composed of the u_n. With the system of coordinates which is formed by the u_n the function f is associated by (5) to a certain *point* of Hilbert space. The coordinates of this point as measured in the system u_n are the expansion coefficients A_n. Hilbert space thus becomes a *function space*. The association between the arbitrary functions and the points of the space of infinitely many dimensions is one-to-one. If we join the point which represents the function f to the origin of the coordinate system of the u_n, then this infinite dimensional vector represents the function f. According to (5a), which we can write in the form $A_n = (f, u_n^*)$, the coordinates of the representative point are the projections of the representative vector on the axes of the system of u_n^*.

From these highly abstract generalizations we return to the physical applications. For the time being we restrict ourselves to the simple problems of acoustics and heat conduction in their historical form. We defer the questions of wave mechanics to the end of this chapter.

The general problem of *acoustics* for the interior of an arbitrary shell S is the following: the wave equation

$$(8) \qquad \frac{\partial^2 v}{\partial t^2} = c^2 \Delta v, \qquad c = \text{speed of sound},$$

is to be solved with the boundary condition $\partial v/\partial n = 0$ so that for $t = 0$ the functions v and $\partial v/\partial t$ become equal to arbitrary prescribed functions v_0 and v_1 in S. This problem is solved by:

$$(9) \qquad v = \sum A_n u_n \cos\omega_n t + \sum B_n u_n \sin\omega_n t,$$

where the A_n and B_n are to be determined so that

$$(9a) \qquad v_0 = \sum A_n u_n \quad \text{and} \quad v_1 = \sum B_n \omega_n u_n$$

Due to the relation of the ω_n with the eigen values k_n (namely $c = \omega_n/k_n$) the second equation can be rewritten as:

$$(9b) \qquad \frac{v_1}{c} = \sum B_n k_n u_n.$$

From this we obtain as in (5a)

$$(9c) \qquad A_n = \int v_0 u_n^* \, d\tau, \quad B_n = \frac{1}{k_n c}\int v_1 u_n^* \, d\tau.$$

We see that this is an *initial value problem;* the *boundary value problem* has been shifted to the u_n.

The general *heat conduction problem* can be solved in the same manner. The difference is that now *one* arbitrary function v_0 suffices to describe the initial state, the initial temperature variation $\partial v/\partial t$ being determined by the differential equation of heat conduction. As a boundary condition we may use any one of the conditions a),b),c) on p. 63, to which we then also subject the eigenfunctions u_n.

We now set

$$(10) \qquad v = \sum A_n u_n e^{-\varkappa k_n^2 t}$$

where $\varkappa$ stands for temperature (not heat) conductivity. The coefficients A_n are again determined by the initial condition $v = v_0$:

$$(10a) \qquad A_n = \int v_0 u_n^* \, d\tau.$$

In addition to this initial condition the function v satisfies the differential equation (25.1) and the boundary condition to which the u_n are subjected.

The potential equation $\Delta u = 0$ has *no* eigenfunctions, or rather every solution which is regular in the interior of S and which satisfies the boundary condition $u = 0$ or $\partial u/\partial n = 0$ must be zero or constant in the interior of S. Hence there can be here no closed "nodal surfaces" $u = 0$ or $\partial u/\partial n = 0$. However, in the next section we shall construct a solution of the general potential boundary value problem (given values $u = U$ on the boundary) from the eigenfunctions of the wave equation.

A solution of the potential equation which is regular in S can also have no maximum or minimum in the interior of S. Extremal values of u can be assumed only on the boundary of S. This follows from Gauss' theorem on the *arithmetic mean* which can be deduced from Green's theorem (see exercise V.2).

Also, *no* eigenfunctions exist for the differential equation $\Delta u - k^2 u = 0$ or the more general $\Delta u - F u = 0$ for positive $F(x,y,z)$ (see exercise II.2).

§ 27. Free and Forced Oscillations. Green's Function for the Wave Equation

The eigenfunctions correspond to free oscillations; in a non-absorbing medium they need no energy supply. We now wish to consider *forced* oscillations, which must be stimulated in the rhythm of their period in order to be able to continue in their purely periodic state. Just as the free oscillations, they are to satisfy a homogeneous surface condition, e.g., $u = 0$; the region S will be assumed to be bounded in the discussions in this section. The measure of stimulation shall, for the time being, be assumed to be a continuous point function in the interior of S, and in analogy to the Poisson equation of potential theory we denote it by ϱ .[3] Correspondingly we write the differential equation of forced oscillations as:

$$\Delta u + k^2 u = \varrho. \tag{1}$$

Here $k = \omega/c$, as we remarked in (26.9a), where ω is the circular frequency of the stimulation and c is the speed of sound. We assume

$$k \neq k_n, \tag{2}$$

i.e., k is different from every eigen value of the region S for the same boundary condition. The case of "resonance" $k = k_n$ will be treated at the end of this section.

According to the Ohm-Rayleigh principle we can expand ϱ in terms of the normalized u_n as in (26.5) and (26.5a):

$$\varrho = \sum A_n u_n, \qquad A_n = \int \varrho\, u_n^*\, d\tau, \tag{3}$$

we also write the solution u of (1) in the same form:

$$u = \sum B_n u_n. \tag{3a}$$

Substituting these expansions in (1) and considering the differential equations $\Delta u_n + k_n^2 u_n = 0$, which differ from (1) and which are satisfied by the eigenfunctions u_n, by equating the coefficients of u_n on both sides we obtain

[3] The function ϱ does not represent charge density as in potential theory, but is of dimension sec^{-1} if u stands for an acoustic velocity potential.

$$B_n = \frac{A_n}{k^2 - k_n^2}, \qquad u = \sum \frac{A_n u_n}{k^2 - k_n^2}. \tag{4}$$

We now consider the special case in which ϱ is a δ-function[4] and hence the stimulation is limited to a simple source point Q of yield 1 (see §10 C). We then have

$$\int_Q \varrho\, d\tau = 1,$$

for a domain of integration which contains the point Q, and

$$\int \varrho\, d\tau = 0.$$

for a domain of integration which does not contain Q. Hence we obtain from (3)

$$A_n = u_n^*(Q) \int_Q \varrho\, d\tau = u_n^*(Q) \tag{4a}$$

and from (4)

$$G(P, Q) = \sum \frac{u_n(P)\, u_n^*(Q)}{k^2 - k_n^2}. \tag{5}$$

where the u of (4) is now denoted by the more suggestive $G(P,Q)$. Indeed this solution is *Green's function of our differential equation* (1) *for arbitrary positions of the action point* P *and the source point* Q *and an arbitrary region* S. We assume only the complete system of eigenfunctions and eigen values for the region S. It should be noted that the Ohm-Rayleigh principle has not been applied to the singular δ-function, but only to the continuous function ϱ of (3), which may, e.g., be taken as a regular Gauss error function. Hence in our derivation we do not need the expansion in terms of the u_n of an arbitrary function but only of certain special everywhere regular functions. In the same manner the termwise differentiation which was needed in the derivation of (4) has been carried out on the regular function (3a) before passage to the limit and not on the limit (5).

Green's function is also the solution of an *integral equation*. In order to demonstrate this we recall equation (10.13a), which holds for every self-adjoint differential expression $L(u)$ and hence in particular for the wave equation $\Delta u + k^2 u$. For the three-dimensional case and the boundary value $u = 0$ it reads:

$$u_Q = \int \varrho\,(P)\, G\,(P, Q)\, d\tau_P. \tag{6}$$

[4] We have dropped the name "peak function" ("Zackenfunktion") which was introduced by the author (see *Jahresber. Deutschen Math. Vereinigung* **21**, 312, 1912) in favor of Dirac's notation " δ-function."

The function $G(P,Q)$ is called the "*kernel*" of the integral equation. Corresponding to the reciprocity theorem d) on p. 1 , which, for complex G, has to be rewritten as

$$G(P,Q) = G^*(Q,P), \tag{6a}$$

we call G a "symmetric kernel." From the structure of (5) we see directly that (6a) is satisfied.

The convergence of the series in (5) is absolute only in the one-dimensional case; in the two- or more dimensional case the convergence is conditioned by the alternation of signs of the eigenfunctions for a suitable arrangement of the series. This is the reason equation (5) does not appear explicitly in Hilbert's theory of integral equations, but in an integrated form in which it converges absolutely. In the one-dimensional case equation (5) has been rigorously proven by Erhardt Schmidt.[5]

The non-absolute convergence of (5) becomes apparent if we try to show by termwise differentiation that the differential equation (1) is satisfied. For then we obtain from the n-th term

$$\Delta u_n + k^2 u_n = \Delta u_n + k_n^2 u_n + (k^2 - k_n^2) u_n = (k^2 - k_n^2) u_n$$

and cancelling the factor $k^2 - k_n^2$ with the denominator and summing with respect to n we obtain

$$\Delta G + k^2 G = \sum u_n(P)\, u_n^*(Q). \tag{6b}$$

For $P = Q$ the sum on the right side consists of positive terms and diverges, as it should; the fact that it converges for $P \neq Q$ and vanishes throughout is caused by the alternating signs and cannot be proven from this representation. The order of increase for $P \to Q$ can be deduced directly from the differential equation (1) as follows. We consider a sphere with small radius r and center Q, and integrate (1) over its interior. Due to the δ-character of ϱ the right side becomes equal to 1. According to Gauss' theorem the first term on the left side becomes

$$\int \frac{\partial G}{\partial r}\, d\sigma = 4\pi r^2 \frac{\partial G}{\partial r},$$

while the second term vanishes. Hence we have

$$\frac{\partial G}{\partial r} = \frac{1}{4\pi r^2}, \qquad G = -\frac{1}{4\pi r} + \text{Const} \quad \text{for } r \to 0. \tag{7}$$

This expresses the fact that $G(P,Q)$ has a *unit source* in the point $P = Q$.

The above formulas can be interpreted best in Hilbert space (see

[5] In his famous dissertation, Göttingen, 1905.

p. 179). Namely, equation (6b) states that $\Delta G + k^2 G$ is the scalar product of the two unit vectors $u(P)$ and $u^*(Q)$. Hence these unit vectors are orthogonal if $u(P)$ and $u(Q)$ are different ($P \neq Q$); if $u(P)$ and $u(Q)$ are equal ($P = Q$) then orthogonality is of course excluded; instead the product becomes infinite. The expression (5) is constructed from the individual terms of the same product with the "resonance denominator" $k^2 - k_n^2$ as weighting factor.

Despite its poor convergence equation (5) has frequently been found useful in wave mechanical computations (see §30). For the time being we apply it in order to close a gap in the theory of spherical harmonics. But first we make a few preparatory remarks:

1. If the system of eigenfunctions is *separable* then the summation in (5) decomposes into three summations corresponding to the three coordinates. For the rectangular solid we should have:

$$\Sigma = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \tag{8}$$

where n,m,l are as in (26.1).

2. Green's function depends only on the position of the points P,Q relative to the boundary surface σ and on their distance R. It is independent of the orientation of the coordinates in space. A transformation of the coordinate system which transforms the surface σ into itself and leaves R fixed leaves $G(P,Q)$ *invariant*.

3. If σ is the surface of a sphere then the condition of invariance is satisfied for every rotation of the *spherical polar system* r, ϑ, φ with $r = 0$ as the center of the sphere. The coordinates r, ϑ, φ shall be those of P, $r_0, \vartheta_0, \varphi_0$ those of Q.

4. In the latter case we face the additional fact that the system of eigenfunctions (26.2) is *degenerate*, since the eigen value k_n as defined by (26.2a) is independent of m. Writing G as a triple sum in analogy to (8), we can take the denominator $k^2 - k_n^2$ and the radial part of the eigenfunctions in front of the summation over m. Hence we have

$$G(P,Q) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Psi_n(k_{nl} r)\, \Psi_n(k_{nl} r_0)}{k^2 - k_{nl}^2}\, Y_n, \tag{9}$$

$$Y_n = \sum_{m=-n}^{+n} \Pi_n^m(\cos\vartheta)\, \Pi_n^m(\cos\vartheta_0)\, e^{im(\varphi - \varphi_0)}. \tag{9a}$$

where Ψ_n stands for the function ψ_n of (26.2) normalized to 1, and Π_n for the spherical harmonic P_n nomalized in the same manner. The function Y_n is a surface spherical harmonic. In (9a) we have used the fact that, due to the real character of Ψ_n and Π_n^m the *conjugate*

complex of the eigenfunction

$$\Psi_n(k_{nl}\,r)\,\Pi_n^m(\cos\vartheta)\,e^{im\varphi}$$

for the argument $Q = (\,r_0, \vartheta_0, \varphi_0\,)$ can be written as

$$\Psi_n(k_{nl}\,r_0)\,\Pi_n^m(\cos\vartheta_0)\,e^{-im\varphi_0},$$

for all values of m between $-n$ and $+n$.

From remark 2 concerning the invariance of G, and from representation (9), we now see that the surface spherical harmonic (9a) has an invariant meaning which is independent of the rotation of the polar coordinate system. But this is the very theorem which we assumed as an axiom for the proof of the addition theorem of spherical harmonics on p. 133. That proof is now completed.

Up to now we have assumed that stimulation of forced oscillation takes place in the *interior* of the region S. We now wish to assume that stimulation takes place from the *surface*. This is the case if, instead of the homogeneous boundary condition $u = 0$, we prescribe the *inhomogeneous* boundary condition

$$u = U. \tag{10}$$

The surface is then held in pulsation with the rhythm ω of the forced oscillation and with the amplitude U which may vary from point to point, while in the interior of S the differential equation (1) holds throughout with $\varrho = 0$. From (10.12) we know that this boundary value problem can be solved with the help of Green's function by the formula

$$u_Q = \int U \frac{\partial G}{\partial \nu_P}\,d\sigma_P, \tag{11}$$

where the variable of integration on the right side is P and the domain of integration is the surface of S ($d\sigma_P =$ element of surface, $d\nu_P =$ element of normal at the point P). According to (5) equation (11) becomes

$$u_Q = \sum \frac{u_n^*(Q)}{k^2 - k_n^2} \int U \frac{\partial u_n(P)}{\partial \nu_P}\,d\sigma_P. \tag{11a}$$

This formula contains the general solution of the famous *Dirichlet problem of potential theory*, for by setting $k = 0$ we obtain

$$u_Q = -\sum \frac{u_n^*(Q)}{k_n^2} \int U \frac{\partial u_n(P)}{\partial \nu_P}\,d\sigma_P. \tag{12}$$

The remarkable fact about this solution is that it is not expanded in

particular solutions of the differential equation $\Delta u = 0$ concerned, but rather in the eigenfunctions of the *wave equation* (there are no eigenfunctions of the potential equation). Equation (12) remains valid if instead of the boundary condition (10) we prescribe the more general condition

$$\frac{\partial u}{\partial n} + h\,u = U$$

except that in this case we must subject the eigenfunctions u_n to the corresponding homogeneous condition

$$\frac{\partial u}{\partial n} + h\,u = 0$$

In the special case of a sphere of radius a we obtain from (12) and the boundary condition (10)

$$\frac{2\pi}{a^2}\,u(r_0, \vartheta_0, \varphi_0) = -\sum_n \sum_l \sum_m \frac{A_{nm}}{k_{nl}} \Psi_n(k_{nl}\,r_0)\,\Psi_n'(k_{nl}\,a)\,\Pi_n^m(\cos\vartheta_0)\,e^{-im\varphi_0}, \tag{13}$$

$$A_{nm} = \iint U\,\Pi_n^m(\cos\vartheta)\,e^{im\varphi}\sin\vartheta\,d\vartheta\,d\varphi. \tag{13a}$$

where Ψ_n and Π_n have the same meaning as before. The extra factor 2π on the left side of (13) is due to the fact that, as with the Bessel functions and the spherical harmonics, we have to normalize the two functions $\exp(-i\,m\,\varphi_0)$ and $\exp(i\,m\,\varphi)$ to 1.

Written in terms of the same variables $Q = (\,r_0, \vartheta_0, \varphi_0\,)$ and expanded in terms of particular solutions of $\Delta u = 0$ our solution reads:

$$2\pi\,u(r_0, \vartheta_0, \varphi_0) = \sum_n \sum_m A_{nm} \left(\frac{r_0}{a}\right)^n \Pi_n^m(\cos\vartheta_0)\,e^{-im\varphi_0}. \tag{14}$$

By comparing these solutions we obtain remarkable summation formulas (see exercise V.3).

Finally, we must consider the exceptional case $k = k_m$. From the mechanics and the electrodynamics of oscillating systems we know the "resonance catastrophe": if the rhythm of the stimulating force equals a proper frequency of the system the oscillations increase to infinity. The condition for this event is $\omega = \omega_m$, and hence $k = k_m$. Equation (1) then assumes the form:

$$\Delta u + k_m^2\,u = \varrho. \tag{15}$$

Here we have an inhomogeneous equation whose left side coincides with the homogeneous equation of a free oscillation.

For simplicity we first consider the two-dimensional case of the membrane of §25, which now, however, is subjected to a periodically changing transversal pressure[6] $\varrho = \varrho(x, y)$ with an arbitrary distribution over the membrane. Do pressure distributions exist for which the resonance catastrophe is avoided, that is, for which equation (15) has continuous solutions throughout (for the boundary condition $u = 0$)? The answer to this question is physically evident: for such a solution the pressure on the membrane may do *no work*. Hence we must have:

$$\int \varrho\, u_m\, d\sigma = 0. \tag{16}$$

The pressure distribution must be orthogonal to the eigenfunction $u = u_m$ with which it is in resonance, e.g., it may have equal magnitude in oppositely oscillating sectors of the membrane; *in particular the pressure along a nodal line may be of arbitrary strength.*

This orthogonality theorem is a corner stone in the theory of integral equations and has important applications in the perturbation theory of wave mechanics. Here we must be content with uncovering its physical basis.

The orthogonality theorem can be adapted directly to the three-dimensional case if in (16) we replace the surface integral with respect to $d\sigma$ by a volume integral with respect to $d\tau$. Then we see that the expansion coefficients A_n and B_n in (3) and (4) vanish for $n = m$. By passing from the continuous distribution ϱ to a δ-function we obtain information about Green's function in the case of resonance. From $A_m = 0$ and equation (4a) we have $u_m^*(Q) = 0$. In other words: *The singularity of Green's function must lie on a nodal surface of the critical proper oscillation u_m.*

For this position and only for this position of the source point Q an everywhere regular Green's function exists. The special form of Green's function for the case of resonance is obtained from the general form (5) by omitting the term involving k_m; it therefore reads:

$$G(P, Q) = \sum_{n \neq m} \frac{u_n(P)\, u_n^*(Q)}{k_m^2 - k_n^2}. \tag{17}$$

§ 28. Infinite Domains and Continuous Spectra of Eigen Values. The Condition of Radiation

With increasing domain the eigen values become closer and closer; for an infinite domain they are dense everywhere; we then deal with a *continuous spectrum of eigen values.*

[6] More precisely: pressure divided by surface tension T (see equation (25.1)). The dimension of ϱ is not that of pressure dyn/cm.², but $\frac{\text{dyn}}{\text{cm}^2} / \frac{\text{dyn}}{\text{cm}} = \text{cm}^{-1}$.

Let us consider, e.g., the interior of a sphere of radius a for vanishing boundary values. For the case of purely radial oscillations its eigen values are given by the equation

$$\psi_0(k_\nu a) = 0, \qquad \psi_0(\varrho) = \frac{\sin \varrho}{\varrho}. \tag{1}$$

Hence $k_\nu a = \nu\pi$ and the difference of successive eigen values is

$$\Delta k_\nu = \frac{\pi}{a} \to 0 \quad \text{for} \quad a \to \infty.$$

We may therefore consider the function $\psi_0(kr)$ which is everywhere regular and vanishes at infinity as an *eigenfunction of infinite space.* Thus, if we have an acoustical or an optical problem in which the prescribed sources are in the finite domain (with a discrete or a continuous distribution), and which is to be solved for a given wave number k, then we can always add the function ψ_0 to the solution. Hence oscillation problems (in contrast to potential problems) are *not* determined *uniquely* by their prescribed sources in the finite domain. This paradoxical result shows that the condition of *vanishing* at infinity is not sufficient, and that we have to replace it by a stronger condition at infinity. We call it the *condition of radiation:* the sources must be *sources*, not *sinks*, of energy. The energy which is radiated from the sources must scatter to infinity; *no energy may be radiated from infinity into the prescribed singularities of the field* (plane waves are excluded since for them even the condition $u = 0$ fails to hold at infinity).

For our special eigenfunctions

$$\psi_0(kr) = \frac{1}{2i}\left(\frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r}\right)$$

the state of affairs is simple: for the time dependence $\exp(-i\omega t)$ e^{ikr}/r is a *radiated*, e^{-ikr}/r an *absorbed*, $\psi_0(kr)$ a *standing* wave (nodal surfaces $kr = \nu\pi$). By excluding absorption from infinity we exclude the addition of the eigenfunction $\psi_0(kr)$. Hence the permissible singularities are restricted to the form

$$u = C\frac{e^{ikr}}{r} \tag{1a}$$

For these singularities we have the condition

$$\lim_{r \to \infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{2}$$

It is called the general *condition of radiation* and we shall apply it to all

acoustic and electrodynamic oscillation problems that are generated by sources in the finite domain.

In fact, condition (2) holds not only for the spherical wave (1a) which radiates from $r = 0$, but it also holds for a stimulation which acts at the point $x = x_0$, $y = y_0$, $z = z_0$

$$u = C\frac{e^{ikR}}{R}, \quad R^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$

Hence, for a continuous stimulation of the spatial density $\varrho = \varrho(x_0, y_0, z_0)$ we have:

$$u = \int \varrho \frac{e^{ikR}}{R}\, dx_0\, dy_0\, dz_0 .$$

This holds not only for unbounded space, but also in the case where there are bounded surfaces σ on which arbitrary linear boundary conditions are prescribed, whether homogeneous, e.g., $u = 0$, or inhomogeneous, e.g., $u = U$. In the former case we have scattered or reflected radiation emanating from the surface σ, whereas in the latter case we have radiation that is stimulated by the pulsating surface σ itself (see p. 186).

As counterpart to the radiation condition (2) we have what may be called the "absorption condition":

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} + iku\right) = 0 . \tag{2a}$$

We demonstrate the general validity of the radiation condition by showing that it guarantees the *uniqueness* of solution of the above general oscillation problem. We may then be convinced that the unique solution of the *mathematical* problem is identical with the *solution that is realized in nature*. Our problem is the following:

a) In the exterior of a surface σ, which may consist of several surfaces $\sigma_1, \sigma_2, \ldots$, the function u is to satisfy the differential equation

$$\Delta u + k^2 u = \varrho$$

The function ϱ measures the yield of the sources which may be continuously distributed or concentrated in single points. The function ϱ is given and must vanish at infinity with sufficient rapidity.

b) On σ the function u is to satisfy $u = U$, where U is a given point function on σ. The surface σ lies entirely in the finite domain.

c) In the finite domain u satisfies the condition (2). The quantity r in (2) stands for the distance from any fixed finite point $r = 0$. Around

this point we draw a sphere Σ of radius $r \to \infty$, which does not intersect the surface σ. The surface element on the sphere is $d\Sigma = r^2\, d\omega$, where $d\omega$ is the solid angle seen from $r = 0$. The region between Σ and σ is called S.

d) Except at possible prescribed sources the function u is to satisfy those conditions of continuity which we prescribed in the derivation of the differential equation.

We assume that two solutions of this problem u_1 and u_2 exist and, as usual, form

$$w = u_1 - u_2, \tag{3}$$

as well as the conjugate function w^*. These functions satisfy the conditions a) to d) with $\varrho = 0$ and $U = 0$. Then in Green's theorem

$$\int_S (w\Delta w^* - w^*\Delta w)\, d\tau = \left\{\int_\sigma d\sigma + \int_\Sigma r^2\, d\omega\right\}\left(w\frac{\partial w^*}{\partial n} - w^*\frac{\partial w}{\partial n}\right) \tag{4}$$

the integral on the left and the first integral on the right vanish. *Hence, the integral over the sphere Σ must also vanish.*

For the further discussion we write:

$$w = \frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{f_n(\vartheta, \varphi)}{r^n}, \tag{5}$$

which is shown to be sufficiently general by the following: we consider w expanded in surface spherical harmonics $Y_n(\vartheta, \varphi)$. According to §24 A the coefficients must be of the form

$$C_n\, \zeta_n^1(k\,r) + D_n\, \zeta_n^2(k\,r)$$

where ζ is connected with the half-index Hankel functions by equation (21.15). But here we must have $D_n = 0$, because of the behavior of ζ_n^2 for large values of the argument (see §21 D, p. 117). At the same place we learned that the ζ_n^1 are composed of a finite number of terms of the form $e^{ikr}/(k\,r)^m$, $m < n$. By arranging this expansion in spherical harmonics according to powers of r^{-n} we obtain (5), where the $f_n(\vartheta, \varphi)$ turn out to be finite sums of surface spherical harmonics.

The f_n satisfy a simple *recursion formula*. According to (22.4) the differential equation $\Delta w + k^2 w = 0$ written in terms of r, ϑ, φ, yields the equation

$$\frac{\partial^2(r\,w)}{\partial r^2} + \frac{1}{r^2}D(r w) + k^2\, r\, w = 0 . \tag{6}$$

where D is the differential symbol of (23.15b) in the coordinates θ, φ. Applying (6) to (5) we obtain

$$e^{ikr}\sum_{n=0}^{\infty}\left(-\frac{2ikn}{r^{n+1}}+\frac{n(n+1)}{r^{n+2}}+\frac{D}{r^{n+2}}\right)f_n=0\,.$$

Replacing the index of summation n in the first term of the parentheses by $n+1$ we obtain

$$e^{ikr}\sum_{n=0}^{\infty}\frac{1}{r^{n+2}}\left[-2ik(n+1)f_{n+1}+\{n(n+1)+D\}f_n\right]=0$$

and hence the recursion formula:

$$2ik(n+1)f_{n+1}=\{n(n+1)+D\}f_n\,. \tag{6a}$$

Hence: *if* $f_0=0$ *then all* $f_1=f_2=\ldots=0$.

We now investigate the remaining integral in (4). Since we are interested in the limit for $r\to\infty$ we can replace w by the first term of its expansion (5), ignoring the higher powers of $1/r$, whence:

$$w=\frac{e^{ikr}}{r}f_0\,,\qquad w^*=\frac{e^{-ikr}}{r}f_0^*;$$

$$\frac{\partial w}{\partial n}=ik\frac{e^{ikr}}{r}f_0\,,\qquad \frac{\partial w^*}{\partial n}=-ik\frac{e^{-ikr}}{r}f_0^*.$$

Thus we obtain:

$$\int r^2\,d\omega\left(w\frac{\partial w^*}{\partial n}-w^*\frac{\partial w}{\partial n}\right)=-2ik\int f_0 f_0^*\,d\omega\,.$$

The integrand is *positive* as long as $f_0\neq 0$. But we saw in (4) that this integral must vanish. Hence

$$f_0=0,\ \text{and due to (6a)}\ f_1=f_2=\ldots=0.$$

Therefore

$$w=0\ \text{and}\ u_2=u_1.$$

The author's original proof[7] of this uniqueness theorem assumed, in addition to the conditions a),b),c) for u, the existence of Green's function for the exterior of the surface and an additional "finality condition". The fact that the latter is superfluous has been rigorously proven by F. Rellich[8] even for the case of an arbitrary number of dimensions h.

[7] See footnote on p. 183 and Frank-Mises II, chap. XIX, §5. The form of the proof given in the text is essentially F. Sauter's.

[8] *Jahresber Deutschen Math. Vereinigung* **53**, 57 (1943), which also treats the case in which the surface σ stretches to infinity.

where the radiation condition reads

$$\operatorname*{Lim}_{r\to\infty} r^{\frac{h-1}{2}}\left(\frac{\partial u}{\partial r}-iku\right)=0. \tag{7}$$

In the two-dimensional case $h=2$, where, as we know, the spherical wave e^{ikr}/r is replaced by the cylindrical wave $H_0^1(kr)$, equation (7) becomes

$$\operatorname*{Lim}_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\partial u}{\partial r}-iku\right)=0, \tag{7a}$$

which actually is satisfied by $u=H_0^1(kr)$. In the one-dimensional case, where the radiating wave is given by $\exp(ik|x|)$, equation (7) becomes

$$\operatorname*{Lim}_{|x|\to\infty}\left(\frac{\partial u}{\partial |x|}-iku\right)=0. \tag{7b}$$

Following Rellich, we stress the fact that no radiating solution u of the wave equation can exist which, in every direction, approaches zero more rapidly than $1/r$. For such a function u we would have $f_0=0$ in (5) and, as we have seen, this causes u to vanish identically. In this respect the wave equation differs from the potential equation. For the latter solutions exist which, for increasing r, decrease more rapidly than $1/r$, the so-called dipole, quadrupole, and octupole fields of §24 C. For the wave equation such an r-dependence, which implies a pole of higher order than $1/r$ at $r=0$, can happen only in the so-called "near zone" ($r<\lambda$, $\lambda=$ wavelength); in the "far zone" ($r>\lambda$) every solution of the wave equation behaves like the spherical wave e^{ikr}/r. Potential theory is the limiting case $\lambda=\infty$, as for this case, the near zone reaches to infinity, so to speak.

We now come to the problem of *Green's function for a continuous spectrum*. We first consider in detail the very simplest one-dimensional example ($-\infty<x<+\infty$), in which the radiation condition is the only boundary condition prescribed. Green's function is then identical with the "principal solution" introduced on p. 47, and therefore has a "unit source" at an arbitrary prescribed point $x=x_0$ (see exercise II.3). It must satisfy the conditions:

a) $$\frac{d^2G}{dx^2}+k^2G=0 \quad \text{for } x\neq x_0$$

b) $$\left(\frac{dG}{dx}\right)_{x_0+0}-\left(\frac{dG}{dx}\right)_{x_0-0}=1, \quad \text{(definition of unit source)}$$

c) $$\frac{dG}{d|x|}-ikG=0 \quad \text{for } x=\pm\infty.$$

The solution is seen to be

$$(8)\qquad G=\begin{cases}\dfrac{1}{2ik}\,e^{ik(x-x_0)} & \text{for } x>x_0\,,\\[2mm] \dfrac{1}{2ik}\,e^{-ik(x-x_0)} & \text{for } x<x_0\,.\end{cases}$$

We compare this to the representation (27.5) first for the finite region $-l<x<+l$, but with the usual boundary conditions replaced by the radiation condition. In preparation for a continuous spectrum we change the name k_n of the eigen values to λ; the eigenfunction $u=u_\lambda$ which belongs to λ is then defined by

$$\text{a)}\qquad \frac{d^2u}{dx^2}+\lambda^2 u=0 \qquad -l<x<+l\,,$$

$$\text{b)}\qquad \frac{du}{d|x|}-iku=0 \qquad |x|=l\,.$$

If we write the solution of a) as:

$$(9)\qquad u=A\,e^{i\lambda x}+B\,e^{-i\lambda x},$$

then according to b) we must have

$$A(\pm\lambda-k)\,e^{\pm i\lambda l}+B(\mp\lambda-k)\,e^{\mp i\lambda l}=0\,,$$
$$\frac{A}{B}=\frac{\lambda+k}{\lambda-k}\,e^{-2i\lambda l}=\frac{\lambda-k}{\lambda+k}\,e^{+2i\lambda l}.$$

From this we obtain the equation for λ:

$$\left(\frac{\lambda-k}{\lambda+k}\right)^2 e^{4i\lambda l}=1\,.$$

This equation splits into the equations

$$(9a)\qquad \frac{\lambda-k}{\lambda+k}\,e^{2i\lambda l}=+1,\quad B=A,\quad u=2A\cos\lambda x,$$

$$(9b)\qquad \frac{\lambda-k}{\lambda+k}\,e^{2i\lambda l}=-1,\quad B=-A,\quad u=2iA\sin\lambda x.$$

From (9a) we obtain as first and second approximation

$$(10a)\qquad \begin{aligned}&\text{for } \lambda\gg k,\ \lambda=\frac{\pi}{l}m \qquad \text{and } =\frac{\pi}{l}m\left(1-\frac{ikl}{\pi^2m^2}\right),\quad m\to\infty\\ &\text{for } \lambda\ll k,\ \lambda=\frac{\pi}{l}\left(m+\tfrac12\right) \text{ and } =\frac{\pi}{l}\left(m+\tfrac12\right)\left(1-\frac{i}{kl}\right),\ m=0,1,2,\ldots\end{aligned}$$

where m is an integer. In the same manner we obtain from (9b)

$$(10b)\qquad \begin{aligned}&\text{for } \lambda\gg k,\ \lambda=\frac{\pi}{l}\left(m+\tfrac12\right) \text{ and } =\frac{\pi}{l}\left(m+\tfrac12\right)\left[1-\frac{ikl}{\pi^2\left(m+\frac12\right)^2}\right],\ m\to\infty,\\ &\text{for } \lambda\ll k,\ \lambda=\frac{\pi}{l}m \qquad \text{and } =\frac{\pi}{l}m\left(1-\frac{i}{kl}\right),\ m=0,1,2,\ldots\end{aligned}$$

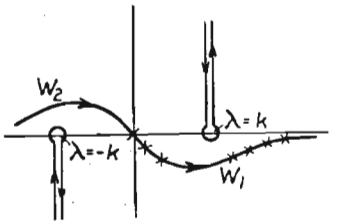

Fig. 26. The path of integration W_1 is completed by the path W_2 to an infinite closed path $W=W_1+W_2$. For positive $x-x_0$ this path can be deformed so that it runs in the positive imaginary λ- half plane.

We see that the values of λ as calculated from (9a) and (9b) form a sequence (marked by x in Fig. 26) that, starting with $\lambda=0$, first descends linearly into the negative imaginary λ- half plane[9] and finally for large λ (large m) osculates the real λ- axis from below. According to (9a,b) the successive points alternatingly belong to cosine and sine eigenfunctions. After normalization to 1 these eigenfunctions are

$$(11)\qquad u=\begin{cases}\dfrac{1}{\sqrt{l(1+\Lambda)}}\cos\lambda x\,,\\[2mm] \dfrac{1}{\sqrt{l(1-\Lambda)}}\sin\lambda x\,,\end{cases}\qquad \Lambda=\frac{\sin\lambda l\cos\lambda l}{\lambda l}\,.$$

In the limit $l\to\infty$ the λ- points of Fig. 26 will be everywhere dense on the right half curve denoted by W_1. The difference between two successive points of the sequence (10a) or (10b) then always becomes

$$(11a)\qquad d\lambda=\frac{\pi}{l}\to 0\,.$$

We now return to the representation (27.5) of Green's function. For $u(P)$ and $u(Q)$ we substitute their expression (11) in the variables x and x_0 respectively, and combine the pairs of successive cosine and sine terms, i.e., the terms which belong to successive eigen values λ. The numerator of (27.5) then becomes

$$u(P)u^*(Q)=\frac{\cos\lambda x\cos\lambda x_0}{l(1+\Lambda)}+\frac{\sin\lambda x\sin\lambda x_0}{l(1-\Lambda)}\,.$$

According to (11) and (11a) we have for $l\to\infty$

[9] The fact that the eigen values are complex in contrast to the theorem on p. 169 is due to the fact that our present boundary condition is of a complex nature.

$$\Lambda = 0, \qquad \frac{1}{l} = \frac{d\lambda}{\pi}.$$

Hence the numerator in (27.5) becomes $\cos\lambda(x-x_0)\,d\lambda/\pi$, while the denominator in our present notation is $k^2-\lambda^2$. Equation (27.5) then becomes

$$G = \frac{1}{\pi}\int_{W_1} \frac{\cos\lambda(x-x_0)}{k^2-\lambda^2}\,d\lambda = \frac{1}{2\pi}\int_{W} \frac{e^{i\lambda(x-x_0)}}{k^2-\lambda^2}\,d\lambda. \tag{12}$$

where W in the last term is the path $W_1 + W_2$ of Fig. 26. The fact that the integration over W_1 is equal to one half the integral over the whole path W follows from the fact that in the integral over W_1 both the numerator and the denominator are even functions of λ. The fact that in the last term we can replace the cosine by the exponential function follows from fact that the sine part of the exponential function is odd in λ and hence vanishes upon integration. The path W is much more convenient than W_1 since it can be deformed away from the origin by the methods of complex integration.

The manner in which this deformation should be performed can be seen from Fig. 26. For positive $x - x_0$ the path W can be drawn over into the positive imaginary λ half plane, for negative $x - x_0$ it can be drawn into the negative imaginary half plane. In the one case the path can not be transformed across the pole $\lambda = +k$ of the integrand in (12), in the other case it can not be transformed across the pole $\lambda = -k$. Forming the residues and combining the two cases we obtain from (12):

$$G = \frac{1}{2ik}e^{ik|x-x_0|}. \tag{13}$$

This is exactly the same as (8).

Hence we see: *The general representation (27.5) of Green's function remains valid for a continuous eigen value spectrum if, in accordance with the radiation condition, we consider a complex path of integration.* If instead we have the "*absorption condition*" (i replaced by $-i$ in (1a) and (2)), then instead of W we have to consider its reflected image on the real λ-axis; we then obtain equation (13) with i replaced by $-i$.

If instead of the one-dimensional case we consider the two- or three-dimensional case and correspondingly replace the coordinate x by the polar coordinates r, φ and r, ϑ, φ, then the spectrum of the eigen values becomes *continuous only in the r-coordinate* but remains *discontinuous* in the angle coordinates. For example in the case of unbounded

three-dimensional space we start from the following representation of Green's function

$$2\pi\,G(P,Q) = \sum_{n}\sum_{m} \Pi_n^m(\cos\vartheta)\,\Pi_n^m(\cos\vartheta_0)\,e^{im(\varphi-\varphi_0)}\int_{W_1}\frac{F\,d\lambda}{k^2-\lambda^2}, \tag{14}$$

$$F = \Psi_n(\lambda r)\,\Psi_n(\lambda r_0). \tag{14a}$$

Here, as in the preceding section, Π and Ψ stand for the spherical harmonic and Bessel functions normalized to 1; and in the following the Z^1, Z^2 correspond to the Hankel functions ζ^1, ζ^2. The factor 2π on the left side is due to the normalization of the functions $\exp\{im\varphi\}$ and $\exp\{-im\varphi_0\}$. As in Fig. 26 the path W_1 lies in the complex λ-plane from $\lambda = 0$ to $\lambda = \infty$. and again avoids the pole $\lambda = k$. We first give a brief discussion of the way in which this representation can be treated in analogy to the one-dimensional case. This will yield a representation of spherical and cylindrical waves which we have met before.

In order to transform W_1 into the path W of Fig. 26, we write

$$\Psi_n(\lambda r) = \tfrac{1}{2}(Z_n^1(\lambda r) + Z_n^2(\lambda r)),$$

For the convergence problems which arise in connection with the normalization we refer the reader to Appendix I. Due to the properties of Hankel functions (see exercise IV.2, in particular equation (12), and also the discussion in connection with equation (32.13)), we can transform the integral over W_1, which involves the function F of (14a), into the integral over W involving

$$F_1 = \tfrac{1}{2}Z_n^1(\lambda r)\,\Psi_n(\lambda r_0) \qquad r > r_0, \tag{14b}$$

and

$$F_2 = \tfrac{1}{2}\Psi_n(\lambda r)\,Z_n^1(\lambda r_0) \qquad r < r_0 \tag{14c}$$

Since the integrand $\frac{1}{2}F_{1,2}/(k^2-\lambda^2)$ vanishes at infinity in the positive imaginary λ-plane for both cases $r \gtrless r_0$, the integral of (14) reduces to the residue at the pole $\lambda = k$:

$$\int F_{1,2}\frac{d\lambda}{k^2-\lambda^2} = \frac{\pi}{2ik}\begin{cases} Z_n^1(kr)\,\Psi_n(kr_0) & r > r_0, \\ \Psi_n(kr)\,Z_n^1(kr_0) & r < r_0. \end{cases} \tag{15}$$

Applying the addition theorem of spherical harmonics (22.34) we obtain from (14):

$$(16)\quad G(P,Q)=\frac{1}{4ik}\sum_{n=0}^{\infty}\Pi_n(\cos\Theta)\,\Pi_n(1)\begin{cases}Z_n^1(kr)\,\Psi_n(kr_0) & r>r_0\\ \Psi_n(kr)\,Z_n^1(kr_0) & r<r_0\end{cases}$$

For reasons of symmetry $G(P,Q)$ in unbounded space is a pure function of the distance

$$R=\sqrt{r^2+r_0^2-2\,r\,r_0\cos\Theta},$$

between P and Q; namely, due to the definition of the unit source on p. 47 we have

$$(16\,a)\qquad G(P,Q)=-\frac{1}{4\pi}\frac{e^{ikR}}{R}=\frac{1}{4\pi i}\zeta_0^1(kR),$$

where ζ_0 is given by (21.15a). If, on the right side of (16), we pass from Π,Ψ,Z to P,ψ,ζ (see Appendix I equation (9a)), we obtain the addition theorem (24.9a).

The corresponding series for the two-dimensional case are contained in (21.3).

More important than the derivation of these known formulas is the generalization to the case in which space is not unbounded but is bounded by a finite closed surface σ (or, in the two-dimensional case, by a curve s) with prescribed boundary conditions. We are then dealing with the *proper problem of Green's function:* to find a function $G(P,Q)$ having a unit source in Q, satisfying the radiation condition at infinity and the given boundary condition on σ (or s).

We choose the special case in which the surface σ is a sphere $r=a$, and the boundary condition is

$$(17)\qquad u=0.$$

The point Q is to lie on the ray

The eigenfunction which belongs to the eigen value λ is no longer $\psi_n(\lambda r)$, but can be written in the (non-normalized) form

$$(18)\qquad u_n(\lambda,r)=\psi_n(\lambda r)+A\,\zeta_n^1(\lambda r)$$

Due to (17) the function A becomes[10]

$$(18a)\qquad A=-\frac{\psi_n(\lambda a)}{\zeta_n^1(\lambda a)}.$$

For the construction of Green's function we shall not follow the general method of equation (14). Instead we shall use a shorter though

[10] The fact that A depends on λ made it necessary to write $u(\lambda,r)$, instead of $u(\lambda\, r)$.

less systematic approach based on equation (24.9) for unbounded space:

$$(19a)\,(19b)\qquad \frac{e^{ikR}}{ikR}=\sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\vartheta)\begin{cases}\psi_n(kr_0)\,\zeta_n^1(kr) & r>r_0,\\ \zeta_n^1(kr_0)\,\psi_n(kr) & r<r_0.\end{cases}$$

Here (19b) will not yet satisfy condition (17) for $r=a$; in order to satisfy (17) we complete the right side of (19b) by adding

$$-\sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\vartheta)\,\zeta_n^1(kr_0)\,\psi_n(ka)\frac{\zeta_n^1(kr)}{\zeta_n^1(ka)},$$

Due to (18) the right side of (19b) becomes

$$(20)\qquad \sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\vartheta)\,\zeta_n^1(kr_0)\,u_n(k,r).$$

If we make the same adjunction to (19a) then the continuous passage from (19a) to (19b) for $r=r_0$ is preserved, as is the radiation condition for $r\to\infty$. The right side of (19a) becomes

$$(21)\qquad \sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\vartheta)\,\zeta_n^1(kr)\,u_n(k,r_0).$$

From (20) and (21) we obtain Green's function by adjoining the factor $k/4\pi i$ which, as in (16a), is due to the condition of a unit source. We then have:

$$(22)\quad G(P,Q)=\frac{k}{4\pi i}\sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\vartheta)\begin{cases}\zeta_n^1(kr)\,u_n(k,r_0) & r>r_0,\\ \zeta_n^1(kr_0)\,u_n(k,r) & r<r_0.\end{cases}$$

This way of writing reveals the connection with our general method in (14). The function F of (14a) is now represented by

$$F=u_n(\lambda,r)\,u_n(\lambda,r_0);$$

except for a constant normalizing factor. The corresponding functions F_1,F_2 of (14b,c) become

$$F_{1,2}=\frac{1}{2}\begin{cases}\zeta_n^1(\lambda r)\,u_n(\lambda,r_0) & r>r_0,\\ \zeta_n^1(\lambda r_0)\,u_n(\lambda,r) & r<r_0\end{cases}$$

By forming the residues for $\lambda=k$ we then obtain equation (22).

In Appendix II of this chapter we shall introduce a novel method of constructing Green's function, which not only improves the convergence

of the series in the most important cases, but also reveals new aspects of the method's applicability.

In the appendix to the following chapter we shall further show that this method would solve the problem of wireless telegraphy on a spherical earth (for infinitely conductive soil and a vertical "dipole antenna") if it were not for the decisive role of the ionosphere.

Finally we remark: a representation of the form (14) remains valid if as the surface σ we choose an ellipsoid instead of a sphere. Instead of the r, ϑ, φ we then have to use the coordinate system of confocal ellipsoids and hyperboloids. The spectrum of eigen values for the exterior of the ellipsoid will then remain discrete in the parameters of the one piece and two piece hyperboloids, but becomes continuous in the parameter of the ellipsoids. By integration over this last parameter we would obtain a simplification similar to that of (22). Even in the most general case where there are no separating coordinates, in which the eigenfunctions can be decomposed into products, we can still use equation (27.5) as a starting point for the representation of Green's function.

§ 29. The Eigen Value Spectrum of Wave Mechanics. Balmer's Term

The Schrödinger equation of wave mechanics for the simple case of the hydrogen atom reads

$$\Delta\psi + \frac{2m}{\hbar^2}(W - V)\,\psi = 0\,. \tag{1}$$

This is our equation (7.15), with the difference that the symbol of energy W has been replaced by the difference of the total energy W and the potential energy V or, mechanically speaking, by the kinetic energy. The Rutherford model for the H-atom consists of a nucleus, the proton with a $+e$ charge, and of an electron with a $-e$ charge that moves in the proton field. Its potential (Coulomb) energy measured in electrostatic units is

$$V = -\frac{e^2}{r}, \tag{2}$$

where r is the distance from the proton and V is normalized so that at infinity we have $V = 0$. The mass energy m_0c^2 of the electron at rest is not to be counted in the total energy. In the following we may consider the proton at rest at the point $r = 0$.

Equation (1) differs from the wave equation we have treated so far because the constant k^2 has been replaced by a *point function* which

becomes *singular* at the point $r = 0$. Whereas we have used k to denote eigen value, we shall now use W as an eigenparameter. Hence, we shall seek those values of W for which (1) has a solution which is continuous in the entire space. These solutions are the eigenfunctions of our "Kepler problem," where the nucleus plays the role of the sun and the electron the role of the planets. Since the electron may move in unbounded space, the spectrum of eigen values will be continuous in the r-coordinate as in equation (28.14). More important for us is the fact that the spectrum also has *discrete* components.

The spectral apparatus gives us the discrete spectrum by measuring the *line spectrum*, which, in the case of hydrogen, is given in the visible range by the Balmer series H_α, H_β, H_γ, H_δ, ... The lines of this spectrum cumulate at the limit given by the Rydberg constant R. The adjoining *continuum* lies in the near ultraviolet range. Both the discrete and the continuous spectrum are given by the Schrödinger equation. This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

Niels Bohr gave a general explanation of the Balmer series and its limiting frequencies twelve years before Schrödinger, by endowing the Rutherford model with certain quantum theoretical traits. However the concept of orbits he used lead to diverse contradictions and had to be abandoned in favor of the analytic model of equation (1). The fact that (1) is also based on quantum theory is indicated by the entrance of Planck's constant $\hbar = h/2\pi$.

What is the physical meaning of the eigenfunction ψ ? The answer to this question shows the complete revolution in the concept of nature that quantum theory has brought about: $|\psi|^2 dx\,dy\,dz$ stands for the *probability* with which we may expect to find the hydrogen electron at the point (x,y,z) within an error of dx,dy,dz. Hence, in wave mechanics the concept of probability takes the place of the concept of strict determinism which rules in classical mechanics. The measure of indeterminacy in the atomic range is Planck's h (Heisenberg).

The "normalization of the eigenfunctions to 1," which so far had been introduced only for mathematical simplicity, thereby acquires a fundamental meaning. Namely, the equation

$$\int |\psi|^2\, d\tau = 1 \tag{3}$$

asserts the *certainty* that the electron is somewhere in space; this condition is necessary from the point of view of wave mechanics. Equation (3) holds for a discrete spectrum; for the continuum it must be modified according to the prescription of Appendix I to this chapter.

We now turn to the integration of (1), introducing the coordinates r, ϑ, φ. If we write the wave equation in the form (22.4) and let[11]

$$\psi = \chi(r)\, P_l^m(\cos\vartheta)\, e^{im\varphi} \tag{4}$$

then according to the differential equation (22.13) we obtain

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left\{\frac{2m}{\hbar^2}\left(W + \frac{e^2}{r}\right) - \frac{l(l+1)}{r^2}\right\}\chi = 0. \tag{5}$$

We first consider the case in which the electron is *tied* to the nucleus. Then W must be negative since the energy of the electron at rest at infinity is normalized to zero. If it is absorbed by the nucleus and stably tied there then its energy is decreased. If, on the other hand, $W > 0$ then even for an infinite distance from the nucleus the electron has positive kinetic energy and, mechanically speaking, has a hyperbolic orbit.

The asymptotic behavior of χ for $r \to \infty$ is obtained from (5) by neglecting all terms with $1/r$ and $1/r^2$:

$$\frac{d^2\chi}{dr^2} + \frac{2m}{\hbar^2} W\chi = 0.$$

For negative W we write

$$\frac{d^2\chi}{d\varrho^2} = \frac{1}{4}\chi, \quad \chi = e^{-\varrho/2}, \quad \varrho = \frac{2r}{\hbar}\sqrt{-2mW}. \tag{5a}$$

The other solution of (5a), namely, $\chi = \exp(+\varrho/2)$, must be neglected since χ is to be finite everywhere.

In order to obtain an exact solution of (5) we write

$$\chi = e^{-\varrho/2}\, v(\varrho) \tag{6}$$

and obtain as the differential equation for v

$$v'' + \left(\frac{2}{\varrho} - 1\right)v' + \left[\frac{n-1}{\varrho} - \frac{l(l+1)}{\varrho^2}\right]v = 0 \tag{6a}$$

with the abbreviation

$$n = \frac{m e^2/\hbar}{\sqrt{-2mW}}. \tag{6b}$$

[11] Here we denote the lower index of P by l instead of n, corresponding to wave mechanical usage: l = azimuthal quantum number, n_r = radial quantum number, $n = n_r + l + 1$ = total quantum number, n = magnetic quantum number, where we now have $-l \leq m \leq +l$.

In order to discuss (6a) we use the method of equation (19.36). We write

$$v = \varrho^\lambda w, \qquad w = a_0 + a_1\varrho + \tag{7}$$

and in analogy with (19.37) we obtain:

$$\lambda(\lambda+1) = l(l+1), \text{ and hence } \lambda = +l. \tag{7a}$$

The other root of (7a) $\lambda = -l-1$ must be excluded, since v as well as χ must remain finite for $\varrho = 0$. The recursion formula for the a_k is obtained in analogy with (19.37a) by equating to zero the coefficients of $\varrho^{\lambda+k-1}$ in the power series obtained from (6a) and (7). Thus we find:

$$a_{k+1}\left[(\lambda+k+1)(\lambda+k) + 2(\lambda+k+1) - l(l+1)\right] + a_k\left[n-1-\lambda-k\right] = 0. \tag{7b}$$

If in this equation we make the coefficient of a_k equal to zero by setting

$$n = k + \lambda + 1, \tag{8}$$

then a_{k+1} vanishes and so do all the subsequent terms in w: *the series breaks off, that is, w becomes a polynomial* of degree k, whose further properties we shall treat later. For the time being we shall stress only the following facts: 1. Due to the factor $\exp(-\varrho/2)$ in (6), we see that as $r \to \infty$ the function χ tends to zero with sufficient rapidity to make possible the normalization of ψ according to (3), no matter what the degree of the polynomial w. 2. If the series did not break off then from (7b) we should obtain an asymptotic behavior of a_k which would make w become infinite to the order $\exp(+\varrho)$ for $\varrho \to \infty$, and the normalization of ψ would be impossible. Hence the requirement that the series for w break off is a wave mechanical *necessity*.

We now consider equation (8). We denote the value of k there by n_r (radial quantum number) and for λ we substitute its value from (7b) (azimuthal quantum number). Hence, according to (8) n is *integral:*

$$n = n_r + l + 1. \tag{8a}$$

This number n is called the "total quantum number." From equation (6b) we obtain:

$$W = W_n = -\frac{m e^4}{2\hbar^2 n^2}. \tag{8b}$$

Setting W equal to the energy quantum $h\nu$ we obtain

(9) $$\nu = \frac{m e^4}{2 h \hbar^2 n^2} = \frac{R}{n^2}$$

where

(9 a) $$R = \frac{2 \pi^2 m e^4}{h^3} .$$

This R is the above mentioned "Rydberg frequency"; it can be measured spectroscopically with extraordinary precision and hence can lead to an improvement of our knowledge of the fundamental constants e, m, h. The number ν of (9) is called the *Balmer term*.

The observable frequency of a spectral line is obtained by the passage of the atom from an initial state 1 to a final state 2 and is computed as the difference of the associated terms ν_2 and ν_1. Hence for the hydrogen spectrum we have

(10) $$\nu = \nu_2 - \nu_1 = R\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right).$$

The Balmer series corresponds to the passage into the final state $n_2 = 2$; the Lyman series which lies in the ultraviolet range corresponds to the passage into the fundamental state of the hydrogen atom $n_2 = 1$; in both cases the passage is from an arbitrary initial state $n_1 > n_2$. Hence we have

(10a) $\nu = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$, $n = 3, 4, 5, \ldots$ Balmer series,

(10b) $\nu = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right)$, $n = 2, 3, 4, \ldots$ Lyman series.

The series with $n_2 = 3$, $n_2 = 4$, ... lie in the infrared domain.

After having learned about the *eigen values* of the H-atom we wish to consider the analytic character of its *eigenfunctions*. With the use of (7), (7a) and (8a) we obtain from (6a)

(11) $$\varrho\, w'' + [2\,(l+1) - \varrho]\, w' + (n - l - 1)\, w = 0.$$

This equation is obtained through $(2l+1)$-fold differentiation from the simpler differential equation

(12) $$\varrho L'' + (1 - \varrho) L' + \mu L = 0 \quad \text{with} \quad \mu = n + l.$$

For every integer μ this equation has one and only one polynomial solution of degree μ. With a suitable normalization we obtain the solutions:

$$\begin{aligned} &\mu = 0, \quad L = 1, \\ &\mu = 1, \quad L = -\varrho + 1, \\ &\mu = 2, \quad L = \varrho^2 - 4\varrho + 2, \\ &\mu = 3, \quad L = -\varrho^3 + 9\varrho^2 - 18\varrho + 6. \\ &\cdots \quad \cdots \quad \cdots \end{aligned}$$

These are precisely the expressions we denoted in exercise I.6 as Laguerre polynomials; equation (12) is the Laguerre differential equation, as indicated by the choice of the letter L. This differential equation coincides with the differential equation (24.29) of the confluent hypergeometric function for the parameters $\alpha = -\mu = -n - l$. Hence we have

(13) $$L = F(-n-l, 1, \varrho) \qquad \text{and} \qquad w = \frac{d^{2l+1} L}{d\varrho^{2l+1}} .$$

Hence from (4), (6), (7) and (7a) we obtain the representation of the hydrogen eigenfunction

(14) $$\psi = N \varrho^l e^{-\varrho/2} \frac{d^{2l+1} L}{d\varrho^{2l+1}} P_l^m(\cos\vartheta)\, e^{im\varphi}.$$

where N is a normalization factor due to (3). From (5a) and (8b) we obtain ϱ :

(14a) $$\varrho = \frac{2r}{na}, \qquad a = \frac{\hbar^2}{m e^2} \sim \frac{1}{2}\, 10^{-8}\ \text{cm}.$$

where, as is customary, a denotes the "hydrogen radius."

In order to justify this notation, and as a single special application of the above, we compute the "probability density" in the "fundamental state" $n = 1$ of the H-atom. For $n = 1$ we have according to (8a) $l = m = 0$, $n_r = 0$ and hence from (14)

$$\psi = -N_1 e^{-\varrho/2} = -N_1 e^{-r/a}, \qquad |\psi|^2 = N_1^2 e^{-2r/a},$$

where from (3) we obtain $N_1 = (\pi a^3)^{-\frac{1}{2}}$. Hence, the probability of finding the electron is distributed spherically over the nucleus. For $r = 0$ this probability assumes its maximum N_1^2, for $r = a$ its value is only $(N_2/e)^2$, but it only vanishes at infinity. The charge density is proportional to this probability. From the point of view of wave mechanical statistics we do not have an electron which is concentrated at a point, but instead we have a *charge cloud* whose principal part is in the interior of a sphere of radius a.

From the older point of view of orbits we must ascribe a *disc-like*

form to the H-atom. The fundamental state (circular orbit of radius a) then corresponds to a circular disc. In a magnetic field all the circular discs of an H-atom gas would have to be parallel to each other and perpendicular to the magnetic force lines; a light ray passing through this gas would have to show "magnetic double-diffraction." Precise measurements by Schütz, though performed not on an H-atom gas but on the analogous Na-vapor, showed no trace of this phenomenon. This is one of the contradictions which have been cleared up by wave mechanics.

A behavior similar to that of the fundamental state of the H-atom is obtained for all states with $l = 0$, the so-called "s-terms" of spectroscopy. For $l = 0$ we obtain from (14)

$$\psi = N_n e^{-\varrho/2} L_n'(\varrho), \qquad \varrho = \frac{2r}{n a}, \qquad n = n_r + 1,$$

which again means spherical symmetry. Such s-terms are the fundamental states of the alkali atoms Li, Na, K, ... The same holds for all completed shells, e.g., the so-called eight-shells of rare gases. The proof is based on the addition theorem of spherical harmonics. This spherical symmetry of the closed shells is obviously of great importance for all chemical applications.

We have to add a few remarks about the continuous spectrum of hydrogen, that is, about the states $W > 0$ (the "hyperbolic orbits" of the older theory). The electron is then no longer tied to the nucleus but is still in the field of the proton.

According to (5a) and (6b) ϱ and n become *purely* imaginary for $W > 0$. In the asymptotic solution (5a) the two signs of ϱ are equivalent; both solutions $\exp\{\pm \varrho/2\}$ can be used. It is unnecessary, and due to the imaginary character of ϱ it is also impossible, to make the series (7) break off. Hence every value of W is permissible. *The W-spectrum becomes continuous* and reaches from $W = 0$ to $W = \infty$. Since, according to (8b) $W = 0$ corresponds to the limit $n = \infty$ of the discrete spectra, we see that to each of these spectra there adjoins a continuous spectrum in the short wave direction. The analytic form of the representation (14) remains valid; but L is now no longer a Laguerre polynomial but a *confluent hypergeometric series which does not break off*, since the parameter $\alpha = -n-l$ in (13) is no longer negative integral but general complex.

§ 30. Green's Function for the Wave Mechanical Scattering Problem. The Rutherford Formula of Nuclear Physics

Nuclear physics originated with Rutherford's experiments on the scattering of α-rays by heavy atoms. Since the electron shells of the

atom are immaterial for the case of α-rays, we may treat the scattering problem in terms of the continuous H-spectrum. We are dealing, in fact, with a *two-body problem:* a *nucleus* (of charge Ze, where Z is the atomic number, $Z = 1$ for the H-spectrum) and a particle interacting with it (in this case an α-particle with mass m_α and charge $Z'e$ where $Z' = 2$; in the preceding case an electron of mass m and charge $-e$ corresponding to the charge number $Z' = -1$). First we want to find that point of the continuous spectrum that corresponds to the energy constant W_α of the incoming α-rays. For an infinite distance between the α-particle and the nucleus the kinetic energy of the α-particle is

$$W_\alpha = \frac{m_\alpha}{2} v^2, \qquad \text{hence} \qquad 2 m_\alpha W_\alpha = (m_\alpha v)^2 = p^2,$$

where $\vec{p} = m_\alpha \vec{v}$ is the kinetic momentum of the α-particle.

If we now pass from the *corpuscular interpretation* of the α-rays to the "complementary" *wave interpretation*, then $p/\hbar$ is, at the same time, the *wave number*[12] k_α of the α-rays. Hence we have

$$k_\alpha = \frac{m_\alpha v}{\hbar} = \sqrt{\frac{2 m_\alpha W_\alpha}{\hbar^2}}. \tag{1}$$

We can, therefore, rewrite the variable ϱ of (29.5a) in the form

$$\varrho = 2 i k_\alpha r. \tag{2}$$

For an arbitrary point of the continuous spectrum (i.e., for an arbitrary value W different from W_α) we replace k_α by λ as in §28. Equations (1) and (2) then generalize to

$$\lambda = \sqrt{\frac{2 m_\alpha W}{\hbar^2}}, \qquad \varrho = 2 i \lambda r. \tag{2a}$$

If, as before, we assume the nucleus at rest then the wave equation (29.1) becomes

$$\Delta\psi + \frac{2 m_\alpha}{\hbar^2}\left(W - \frac{Z Z' e^2}{r}\right)\psi = 0. \tag{3}$$

For the time being we replace (3) by:

[12] In fact the formula $k_\alpha = p/\hbar$ is the equation of L. de Broglie: "h times the reciprocal of the wavelength equals the momentum," which in turn is the relativistic completion of Planck's equation: "h times the reciprocal of the time of oscillation equals the energy."