isogonality can also be proved without great difficulty by remaining in the plane, but we may leave this to the reader.

It is now clear how to define *reflection in an arbitrary circle*: Let $\mathfrak{k}_0$ be the circle with radius r and the point a as center. Then, reflection in $\mathfrak{k}_0$ is understood to be the transition from a point $z \neq a$ in the plane, to that point z' which is on the same ray issuing from a as z is, and is such that the product of the distances $|z - a|$ and $|z' - a|$ is equal to r^2. The reflection of a, however, is again to be regarded as the point ∞. If $\mathfrak{k}_0$ is a straight line, reflection in it shall have the elementary meaning. These somewhat more general reflections have, of course, properties entirely analogous to those possessed by reflection in the unit circle. We prove the following theorem concerning them:

THEOREM. *Let a circle* $\mathfrak{k}$ *and two points* z_1 *and* z_2 *symmetric to it be reflected in a circle* $\mathfrak{k}_0$, *yielding the circle* $\mathfrak{k}'$ *and the points* z_1' *and* z_2', *say. Then* z_1' *and* z_2' *are symmetric with respect to* $\mathfrak{k}'$.

For, any two circles which pass through the points z_1 and z_2 are (by 4f) orthogonal to $\mathfrak{k}$. But then their images $\mathfrak{k}_1'$ and $\mathfrak{k}_2'$ are orthogonal to $\mathfrak{k}'$, because of the circularity and isogonality. They therefore intersect (again by 4f) in two points symmetric with respect to $\mathfrak{k}'$.

18. Mapping by means of arbitrary linear functions

Let there be given, finally, an arbitrary linear function

$$w = \frac{az + b}{cz + d}. \tag{1}$$

Then c and d must not both vanish. If $c = 0$, and hence $d \neq 0$, we have the entire linear function $w = (a/d)z + (b/d)$, whose mapping we are already familiar with. If $c \neq 0$, we can write (1) in the form

$$w = -\frac{ad - bc}{c} \cdot \frac{1}{cz + d} + \frac{a}{c}. \tag{2}$$

From this we infer that (1) is identically constant (and hence, the mapping degenerates) if, and only if, the determinant of the four coefficients is equal to zero.[33] *We therefore assume that*

[33]This is obviously also true for the case $c = 0$.

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by Konrad Knopp

$$ad - bc \neq 0 \tag{3}$$

for all linear functions of the form (1) *appearing in what follows.* Then the mapping furnished by (1) can be obtained in three steps:

1) by the mapping $z' = cz + d$,
2) by the mapping $z'' = 1/z'$, and
3) by the mapping $w = a_1 z'' + b_1$, with

$$a_1 = -\frac{ad - bc}{c}, \qquad b_1 = \frac{a}{c}.$$

The first and third are similarity mappings, the second is the one investigated in §17. We therefore have immediately the following *principal theorem*:

THEOREM 1. (1) *furnishes a one-to-one mapping of the full z-sphere on the full w-sphere. This mapping is isogonal and circular.*

In particular, the point $z = -d/c$ goes over into $w = \infty$ (for, it yields first $z' = 0$, then $z'' = \infty$, and, consequently, also $w = \infty$), and $z = \infty$ goes over into $w = a/c$. It is therefore reasonable to *stipulate*, as a supplement to §17(5), *that, when considering linear functions,*

$$\frac{a \cdot \infty + b}{c \cdot \infty + d} = \frac{a}{c}. \tag{4}$$

The point z, whose image is *preassigned* to be the point w, is given, according to (1), by

$$z = \frac{-dw + b}{cw - a}. \tag{5}$$

The linear function (5) is therefore called the *inverse* of (1). The determinant of the coefficients of (5) is the same as that of (1).

The angle between two curves at ∞ is understood to be, of course, the angle at which they intersect *on the sphere* at the north pole. The meaning of isogonality is also clear, then, if the image point, or its original, or both, lie at ∞. Thus, *when considering linear functions, the point* ∞ *is in no way singled out from the other points to play an exceptional role.*

On the basis of the last theorem in §17, we can state, finally, the following one:

THEOREM 2. *Under every mapping of the form* (1), *the figure of a circle*[24] *and two points symmetric with respect to it goes over into the same kind of figure.*

For, the similarity transformations 1) and 3) certainly possess this property; and 2) likewise, because it is equivalent to the successive performance of two reflections, each of which, according to the Theorem of §17, possesses the property.

[24]The straight lines are to be included here.

CHAPTER V

NORMAL FORMS AND PARTICULAR LINEAR TRANSFORMATIONS

19. The group-property of linear transformations

Let us go from a z-plane to a $\mathfrak{z}$-plane by means of a first linear mapping

$$\mathfrak{z} = \frac{a_1 z + b_1}{c_1 z + d_1} = l_1(z), \tag{1}$$

and thence to a w-plane by means of the linear mapping

$$w = \frac{a_2 \mathfrak{z} + b_2}{c_2 \mathfrak{z} + d_2} = l_2(\mathfrak{z}). \tag{2}$$

Then a simple calculation shows that the direct transition from the z- to the w-plane is effected by the function

$$w = \frac{az + b}{cz + d} = l(z), \tag{3}$$

whose four coefficients can be read off from the "*matrix equation*"

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix}.$$ [25]

By compounding two linear mappings $\mathfrak{z} = l_1(z)$, $w = l_2(\mathfrak{z})$ we thus again obtain a linear mapping

$$l(z) = l_2(l_1(z)) = l_2 l_1(z). \tag{4}$$

If l_1 and l_2 do not degenerate, neither does l. For, according to the multiplication theorem for determinants, or by a simple calculation, we find that

[25]The *rows* of the first matrix are "combined" with the *columns* of the next; i.e., the sum of the products of the corresponding elements of the two is formed,—just as when multiplying determinants.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix},$$

and since neither factor is zero, the product is not zero. It is also easy to verify that this compounding or "symbolic multiplication" of linear functions is associative, i.e.,

$$(5) \qquad l_3(l_2 l_1) = (l_3 l_2) l_1 .$$

Every function has also an inverse; for, according to §18, (5), the inverse of (3) is the function

$$w = \frac{-dz + b}{cz - a}.$$

It is denoted by $l^{-1}(z)$. When compounded with $l(z)$, it yields the identity:

$$ll^{-1}(z) = l^{-1}l(z) \equiv z,$$

which corresponds to the coefficient array

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.^{36}$$

On the basis of these facts, we can state the following

THEOREM. *The linear mappings form a group, if the compounding of linear functions is employed as group multiplication. The identity is the identity element of the group, inverse functions are inverse elements.*

20. *Fixed points and normal forms*

In §17 we already spoke of *fixed points* of a mapping. A fixed point is understood to be one which coincides with its image. If this is to be the case for a point z under the mapping

$$(1) \qquad w = \frac{az + b}{cz + d} = l(z),$$

z must satisfy

[36] Or $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, with $a \neq 0$.

$$(2) \qquad \frac{az + b}{cz + d} = z \qquad \text{or} \qquad cz^2 - (a - d)z - b = 0.$$

This is a quadratic equation in z, whose coefficients all vanish only if the mapping is the identity ($a = d \neq 0$, $b = c = 0$). We therefore have immediately

THEOREM 1. *A linear mapping which is not the identity has at most two fixed points.—Hence, if a linear mapping is known to have at least three fixed points, it must be the identity.*

If $c \neq 0$, so that the mapping is a *fractional* linear one, both fixed points (which, of course, may also coincide) are finite. If $c = 0$, in which case we have an *entire* linear mapping, at least one of the fixed points lies at ∞ (this follows already from §16). If, moreover, $a = d$ (but $b \neq 0$), we are dealing with a translation, which leaves only the point ∞ fixed. Hence, as a supplement to the preceding theorem, we have

THEOREM 2. *The point ∞ is a fixed point if, and only if, the linear mapping is entire; it is the only fixed point if, and only if, the mapping is a translation.*

Through the use of the fixed points, one can acquire an even more vivid insight into the nature of the linear mapping.

1) First, let

$$(3) \qquad w = az + b$$

be an entire linear mapping which is not a translation (consequently $a \neq 1$). In addition to the fixed point ∞, it has the finite fixed point

$$(4) \qquad \zeta = \frac{b}{1 - a}.$$

By using ζ, the mapping (3) can be brought into the form

$$(5) \qquad w - \zeta = a(z - \zeta),$$

from which it can be interpreted as follows: In the z-plane, first perform the translation $z - \zeta$ (it moves the point ζ to the origin). Now effect the rotary stretching (a), and finally bring the point 0 back to ζ by a translation. Thus we see that the mapping (3) signifies simply a rotary stretching (a) with the fixed point ζ as center! The mapping has become perfectly clear through this interpretation. In particular, it is evident that the pencil of straight lines through ζ goes over into itself as a whole, and the same holds for the family of concentric circles about ζ as center: under a pure rotation ($|a| = 1$), each of the circles individually; under a pure stretching ($a > 0$), each of the straight lines individually.

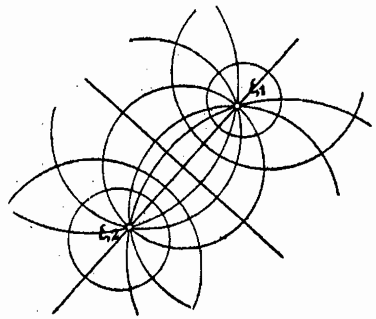

FIGURE 16

2) Now let $c \neq 0$, so that both fixed points, call them ζ_1 and ζ_2, are finite. Moreover, let $\zeta_1 \neq \zeta_2$. Then, it follows immediately from the circularity of the mapping, that the pencil of circles through ζ_1 and ζ_2 goes over into itself as a whole (Fig. 16). More particular information about this can be obtained as follows:

By the mapping

(6) $$\mathfrak{z} = \frac{z - \zeta_1}{z - \zeta_2} = l_0(z),$$

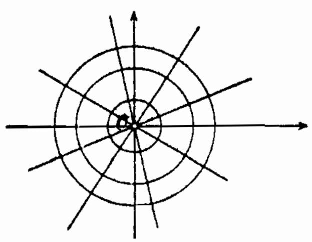

FIGURE 17

which brings ζ_1 to 0 and ζ_2 to ∞, the pencil in Fig. 16 is mapped into the pencil of straight lines in Fig. 17: the circles through 0 and ∞. Let the first lie in the z-plane, the second, in the $\mathfrak{z}$-plane. If we imagine the first to lie in the w-plane, the second, in a $\mathfrak{w}$-plane, then, analogously,

(7) $$\mathfrak{w} = \frac{w - \zeta_1}{w - \zeta_2} = l_0(w)$$

maps Fig. 16 into Fig. 17. Because of the group-property of the linear functions, a linear mapping of the $\mathfrak{z}$-plane on the $\mathfrak{w}$-plane, namely, the mapping

(8) $$\mathfrak{w} = l_0 l l_0^{-1}(\mathfrak{z}),$$

is effected hereby and by the mapping (1). Since (8) has 0 and ∞ as fixed points, it is, according to 1), a rotary stretching with 0 as center, and therefore has the simple form $\mathfrak{w} = \mathfrak{a}\mathfrak{z}$, where $\mathfrak{a}$ is a certain complex number. Consequently, by using the fixed points, (1) can be brought into the following *normal form*:

(9) $$\frac{w - \zeta_1}{w - \zeta_2} = \mathfrak{a}\frac{z - \zeta_1}{z - \zeta_2}.$$

The value of $\mathfrak{a}$ is found immediately by setting $z = \infty$, which yields $w = a/c$. Hence,

(10) $$\mathfrak{a} = \frac{a - c\zeta_1}{a - c\zeta_2}.$$

The argument carried out shows more at the same time: Since there is a family of circles orthogonal to the pencil of straight lines in Fig. 17, namely, the family of circles about 0 as center, there also exists a family of circles orthogonal to the pencil of "circles *through* ζ_1 and ζ_2" in Fig. 16. This family we shall call, for brevity, the family of "circles *about* ζ_1 and ζ_2." It, too, and hence the complete Fig. 16, goes over into itself as a whole, under the mapping (1). Beyond this we can say more precisely:

a) If $|\mathfrak{a}| = 1$, then $\mathfrak{w} = \mathfrak{a}\mathfrak{z}$ is a pure rotation, and, consequently, each of the circles of the second family goes over into itself as a whole, while those of the first family are permuted among themselves. The mapping (1) is then said to be *elliptic.*

b) If $\mathfrak{a}$ is real and positive, it is just the reverse. The mapping (1) is then said to be *hyperbolic.*

c) If $|\mathfrak{a}| \gtrless 1$ and $\mathfrak{a}$ is not a positive real number, the mapping is said to be *loxodromic.* It is obtained by carrying out steps a) and b) in succession.[37]

3) Let $c \neq 0$ once more, but now suppose that $\zeta_2 = \zeta_1$. The mapping is then said to *parabolic.*[38] The totality of circles through the fixed point—call it ζ—goes over into itself as a whole. A family of circles through ζ, which have there a com-

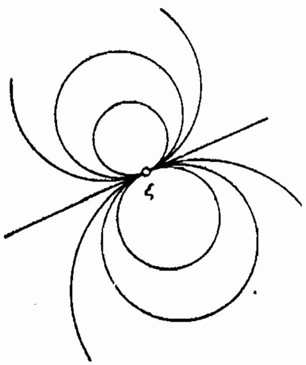

FIGURE 18

mon tangent (see Fig. 18), likewise remains unchanged as a whole. This, as well as further details, are again seen more clearly if the fixed point (in the z-plane *and* in the w-plane) is sent to ∞ by the auxiliary mappings

$$(11) \qquad \mathfrak{z} = \frac{1}{z - \zeta}, \qquad \mathfrak{w} = \frac{1}{w - \zeta},$$

respectively. The circles in question then go over into the straight lines of the plane; those circles having a common tan-

[37] If the auxiliary plane of Fig. 17 is used, the individual steps can be followed even more closely.

[38] The same terminology, of course, is used for *entire* linear mappings: a translation is said to be *parabolic*; a rotary stretching with the fixed point as center (cf. (5)) is said to be *loxodromic*; a pure stretching, *hyperbolic*; a pure rotation, *elliptic.*

gent, into a family of parallels. Again $\mathfrak{z}$-plane and $\mathfrak{w}$-plane are mapped linearly on each other. But since ∞ is the only fixed point under this mapping, the two planes arise from each other by means of a *translation* $\mathfrak{w} = \mathfrak{z} + \mathfrak{b}$. Hence, in the parabolic case, (1) can be brought into the form

$$(12) \qquad \frac{1}{w - \zeta} = \frac{1}{z - \zeta} + \mathfrak{b};$$

and since $z = \infty$ and $w = a/c$ correspond, we must have

$$(13) \qquad \mathfrak{b} = \frac{c}{a - c\zeta}.$$

The family of circles lying in the z-plane, passing through ζ, and having there a common tangent, is transformed by (11) into a family of parallels in the $\mathfrak{z}$-plane. This last family is left unchanged, as a whole, by the translation ($\mathfrak{b}$), and hence, returns, by virtue of (11), to the initial family.

21. *Particular linear mappings. Cross ratios*

Theorem 1 of §18, in particular, the fact that circles are always transformed into circles by linear mappings, remains the most important of all that precedes. We shall now investigate more closely how this takes place. To this end we first prove

THEOREM 1. *Three given distinct points z_1, z_2, z_3 can always be carried into three prescribed distinct points w_1, w_2, w_3 by one, and only one, linear mapping, $w = l(z)$.*[39]

Proof: The equation

$$(1) \qquad \frac{w - w_1}{w - w_3} : \frac{w_2 - w_1}{w_2 - w_3} = \frac{z - z_1}{z - z_3} : \frac{z_2 - z_1}{z_2 - z_3}$$

defines a definite linear function $w = l(z)$. For, on the left we have a linear function of w, on the right, a linear function of z. If we call these $l_1(w)$ and $l_2(z)$, respectively, then $w = l(z) = l_1^{-1}l_2(z)$. Here we must agree, in accordance with §18 (4), that if one of the points z_ν or w_ν is the point ∞, the quotient of those two differences which contain this point is to be replaced by 1.

[39] One of the points z_ν, as well as one of the w_ν, ($\nu = 1, 2, 3$), may also be the point ∞.

This function $w = l(z)$, now, accomplishes the desired end. For, $l_2(z)$ assumes, for $z = z_1, z_2, z_3$, the respective values 0, 1, ∞, and $l_1(w)$ takes on these values for $w = w_1, w_2, w_3$, respectively. Hence, $l(z_\nu) = w_\nu$, ($\nu = 1, 2, 3$). If the linear function $w = L(z)$ accomplishes the same, then the linear function $L^{-1}l(z)$ obviously has the three distinct fixed points z_1, z_2, z_3, and, consequently, according to §20, Theorem 1, is the identity. Therefore $L(z) = l(z)$. This completes the proof.

Now, an oriented circle (including the oriented straight line) is uniquely determined by three (distinct) points given in a definite order. Hence, from Theorem 1 follows immediately the further

THEOREM 2. *A given oriented circumference of a circle in the z-plane or on the z-sphere can always be mapped by one, and only one, linear function, into a given oriented circumference of a circle in the w-plane, in such a manner, that three given points of the z-circle thereby go over into three given points of the w-circle, provided that on both circles the points succeed one another in the sense of the orientation.*

The complex plane is divided by a circle (or a straight line) into two parts. That one of them which lies *to the left* of the orientation will be called "*the interior*" of the circle, and the other one, "*the exterior*" of the circle.[40] The complex sphere is divided by a circle into two spherical caps. That one which, viewed inside the sphere, lies *to the left* of the orientation we shall call *the interior* of the circle, the other, *the exterior* of the circle, so that interiors of circles correspond to interiors of circles under stereographic projection. Since the mapping of full spheres by means of linear functions is one-to-one and, moreover, isogonal without reversion of angles, there follows, now, as a supplement to Theorem 2:

THEOREM 3. *The linear function mentioned in Theorem 2 maps the interior of the z-circle in a one-to-one manner on the interior of the w-circle; and likewise, naturally, the exterior of the first on the exterior of the second.*

To express briefly the hereby established property of linear

[40]Thus, if, e.g., the axis of imaginaries is oriented from bottom to top, the left half-plane is the interior, the right half-plane is the exterior.

functions, that if they map two oriented circles into one another they also map their interior regions (and likewise their exterior regions) on one another in a one-to-one manner, we say that *regions are preserved* under linear mappings.

Examples of these, and the mappings mentioned later on, follow in §22.

The peculiar expressions appearing in (1) are called *cross ratios*. The following is a more precise

DEFINITION. *Let z_1, z_2, z_3, z_4 be four distinct points on the sphere. Then the expression*

$$(z_1, z_2; z_3, z_4) = \frac{z_4 - z_1}{z_4 - z_3} : \frac{z_2 - z_1}{z_2 - z_3} \tag{2}$$

shall be called their cross ratio. If one of the points lies at ∞, then the agreement made above comes into force.[41]

From the proof of Theorem 1 now follows immediately

THEOREM 4. *The cross ratio of four points remains invariant under linear mappings.*

That is to say: If the four points z_ν go over into the respective points w_ν, ($\nu = 1, 2, 3, 4$) under the mapping $w = l(z)$, then

$$(w_1, w_2; w_3, w_4) = (z_1, z_2; z_3, z_4).$$

For, since $w = l(z)$ performs what is required in Theorem 1, it must be the function given by (1). It also carries z_4 into w_4. Hence, for $z = z_4$, $w = w_4$, (1) immediately yields the assertion.

An oriented circle can be given by means of *one* point of the circumference and a pair of points symmetric with respect to the circle, instead of by means of three points of the circumference. This yields, in connection with Theorem 1,

THEOREM 5. *An oriented z-circle can always be linearly*[42]

[41]The order in which the four points are taken is not essential, but, of course, once it has been chosen, it must be retained. If the four points are permuted in all possible ways, we do not obtain 24 distinct values of the cross ratio, but, on the contrary, at most 6. If one of the values is equal to δ, the others are $1/\delta$, $1 - \delta$, $1/(1 - \delta)$, $\delta/(\delta - 1)$, and $(\delta - 1)/\delta$. These values may coincide in part.

[42]It can be shown that *the most general function which maps the interior of one circle in a one-to-one and conformal manner on the interior of another circle, is a linear function.* See, e.g., L. R. Ford, *Automorphic Functions*, New York, 1929, p. 32.

mapped, in one, and only one, way, into an oriented w-circle in such a manner, that a given boundary point z_1 and a given interior point z_0 of the z-circle thereby go over into correspondingly situated, prescribed points w_1 and w_0.

Proof: Let z_0', w_0' be the reflections of z_0, w_0 in their respective circles. Then, according to §18, Theorem 2, a linear function which accomplishes what is required must also carry z_0' into w_0'. Hence, only the linear function resulting from

$$(3) \qquad (w_1, w_0; w_0', w) = (z_1, z_0; z_0', z)$$

can accomplish the desired end, and, according to the preliminary remark, this is also the case.

22. *Further examples*

1. *Mapping the upper half-plane* (*UH*) *on the unit circle* (*UC*).

a) If we require, say, that the interior point i of the UH go over into the center of the UC, and that the boundary point 0 of the UH go over into the boundary point -1 of the UC, then, according to §21, Theorem 5, the mapping is uniquely determined. Since it sends $-i$ to ∞, the points $z = i, 0, -i$ go over into $w = 0, 1, \infty$, respectively.[43] Hence, according to §21, Theorem 1,

$$\frac{w-0}{-1-0} = \frac{z-i}{z+i} : \frac{0-i}{0+i} \quad \text{or} \quad w = \frac{z-i}{z+i}$$

…equired mapping. For every real z, $|w| = 1$, as is easily … Conversely, by means of the inverse function

$$z = -i\,\frac{w+1}{w-1},$$

n…
w…
to … …plane is mapped on the upper z-half-plane.
T… …f the mapping become more vivid if we
the i… …e UH go over into the radii of the UC,
of the … into the circles which have the same
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To e…

[43]Thus, … …ts in such a manner, that $w_3 = \infty$. For
the left hal… …e form $(w - w_1)/(w_2 - w_1)$, and thus
… …we must finally solve, only in the

center as the UC but are smaller than the latter. Because of the circularity of the mapping, we immediately obtain: Referring to the circles in the z-plane which pass through $+i$ and $-i$, those parts of them which lie in the UH are transformed into the radii of the UC; and the circles "about $+i$ and $-i$" which are orthogonal to the first pencil of circles and lie in the UH are transformed into the circles which have the same center as the UC but are smaller than the latter.

By means of this mapping, moreover, the first quadrant of the z-plane is mapped on the lower half of the UC; thus, a quarter-plane, on a semicircle.[44]

b) If we require, somewhat more generally, that the point z_0, ($\Im(z_0) > 0$), in the UH go over into the origin, and that the boundary point α, (α real), go over into the boundary point -1, then

$$(3) \qquad w = -\frac{\alpha - \bar{z}_0}{\alpha - z_0}\cdot\frac{z - z_0}{z - \bar{z}_0} = c\,\frac{z - z_0}{z - \bar{z}_0}, \qquad (|c| = 1),$$

accomplishes what is required. The further details are entirely similar to those under a).

c) The requirement that the three boundary points 0, 1, ∞ of the UH go over into the boundary points i, -1, $-i$ of the UC, also determines the mapping uniquely. We find (and it can be verified subsequently by substituting the z-values):

$$(4) \qquad w = -i\,\frac{z-i}{z+i}.$$

2. *The UH of the z-plane is to be mapped on the UH of the w-plane in such a manner, that the points $z = \infty$, 0, 1 go over into the points $w = 0$, 1, ∞, respectively.* The mapping is hereby uniquely determined. Proceeding as in 1., we find

$$(5) \qquad w = -\frac{1}{z-1}.$$

[44]It is earnestly recommended that the reader make simple sketches for all of the mappings discussed, letting corresponding points and parts of boundaries or regions become clear by using the same colors or hatchings.

Under this mapping, what becomes of the parallels to the axis of reals; the parallels to the axis of imaginaries; the two quadrants of the half-plane? What are the answers to the "inverse" questions?

3. *The exterior of the UC is to be mapped on the right half-plane.* If we send, let us say, the points

$$z = 1, -i, -1 \qquad \text{to} \qquad w = i, 0, \infty,$$

respectively, then in each of the planes the region in question lies to the left of the orientation given in this manner. Hence, the mapping

$$w = i + \frac{z-1}{z+1} = \frac{(1+i)z + (-1+i)}{z+1} \tag{6}$$

does what is required. We leave it to the reader to determine what becomes of the circles which have the same center as the *UC* but are larger than the latter, and what becomes of those parts of the rays issuing from 0, which lie outside the *UC*.

4. *The UC is to be mapped on itself* in such a manner, that the interior point z_0, ($|z_0| < 1$), is transformed into the center.

If z_0 is to go over into 0, the reflection of z_0 in the unit circle, i.e., the point $z_0' = 1/\bar{z}_0$, must be sent to ∞. Hence, the linear function we are looking for must have the form

$$w = c\,\frac{z - z_0}{z - (1/\bar{z}_0)} \qquad \text{or} \qquad w = c'\,\frac{z - z_0}{\bar{z}_0 z - 1}.$$

Now, the radius of the image circle will again be equal to 1 if, and only if, the image of the point $+1$ has the absolute value 1:

$$\left| c'\,\frac{1 - z_0}{\bar{z}_0 - 1} \right| = |c'| = 1.$$

Hence, in particular, the function

$$w = \frac{z - z_0}{\bar{z}_0 z - 1}$$

yields the required mapping.

5. *Two circles which have no point in common can always be transformed, by a linear mapping, into two concentric circles.* For,

two circles of the first kind (whether one encloses the other or not) can always be regarded (in precisely one way) as two circles of a pencil of the kind described in connection with Fig. 16; i.e., there is precisely one pair of points ζ_1 and ζ_2 such that the given circles belong to the pencil of "circles about ζ_1 and ζ_2."[45] The mapping

$$w = \frac{z - \zeta_1}{z - \zeta_2}$$

then obviously performs what is required.

Finally, we prove the following *Theorem:*

6. *The cross ratio of four points is real if, and only if, the points lie on a circle (or a straight line).* For if the cross ratio is to be real,

$$\operatorname{am} \frac{z_4 - z_1}{z_4 - z_3} \qquad \text{and} \qquad \operatorname{am} \frac{z_2 - z_1}{z_2 - z_3}$$

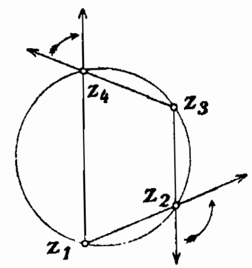

FIGURE 19

must either be equal or differ only by $\pm\pi$. The first amplitude signifies the angle through which the direction extending from z_3 to z_4 must be rotated in the positive sense until it coincides with the direction leading from z_1 to z_4, and the second amplitude has an analogous meaning. Elementary theorems on peripheral angles now establish the validity of the theorem (see Fig. 19),—irrespective of whether the pair of points z_2, z_4 are separated by the pair z_1, z_3 or not.

[45]The "base points" ζ_1 and ζ_2 of the pencil are found by drawing the line of centers of the circles. ζ_1 and ζ_2 then separate harmonically the pair of points of intersection of each of the circles with this line.