FIRST ORDER, LINEAR O.D.E.

$$\frac{dy}{dx} + p(x)\,y = r(x) \qquad ;\ p, r \text{ continuous}$$

usually: ; p, r continuous

let $P(x) = \int_{x_0}^{x} p(\xi)\,d\xi$

$$\therefore\ y = e^{-P(x)} \int_{x_0}^{x} e^{P(\xi)}\, r(\xi)\,d\xi + \underset{\overset{\shortparallel}{y_0}}{A}\, e^{-P(x)}$$

if $y(x_0) = y_0$

Derivation: <u>integrating factor</u>

$$e^{P}(y' + \underset{\overset{\shortparallel}{P'}}{p}\,y) = \frac{d}{dy}(e^{P} y) = e^{P} r$$

Or: homogeneous: $dy/dx + py = 0 \Rightarrow y = u_0 e^{-P(x)} \rightarrow u(x) e^{-P(x)}$

(<u>variation of parameters</u>: let u vary to satisfy inhom. equ.)

substitute: $u' e^{-P} - \cancel{p u e^{-P}} + \cancel{p u e^{-P}} = r \Rightarrow u' = e^{P} r \Rightarrow$

$$u = \int_{x_0}^{x} r e^{P}\, d\xi + A$$

* What if p, r discontinuous: e.g. piecewise continuous

  - Solve eqn. in each interval where p, r cont.
    Solution contains integration constant
  - Choose constants to ensure continuity of solution at jump points for p, r.

* Singular point: $p(x_0) = \infty$ Variety of behaviors:

May be impossible to satisfy $y(x_0) = y_0$ / or if possible, non-unique

Ex: $y' + y/x^3 = 0 \Rightarrow y = A e^{1/2x^2}$. If $A \neq 0$, $y(0^+), y(0^-)$ ∞.

If we demand $|y(0)| < \infty \Rightarrow A = 0 \Rightarrow y \equiv 0$.

Ex: $y' - \nu\, y/x = 0$, $\nu > 0$: $y = A x^{\nu}$; impossible $y(0) \neq 0$.

If we allow $y(0) = 0 \Rightarrow$ nonunique solution.

To understand behavior at singular points: use complex variables.

* If $p(z)$, $r(z)$ analytic in simply conn. D.

Then $\frac{dw}{dz} + p(z)\,w = r(z)$

determines $w(z)$ as analytic function.

Given $w(z_0) = w_0$, unique solution

$$w(z) = w_0 e^{-P(z)} + e^{-P(z)} \int_{z_0}^{z} r(\zeta)\, e^{P(\zeta)}\, d\zeta$$

$$P(z) = \int_{z_0}^{z} p(\zeta)\, d\zeta$$

integrals independent of path by Cauchy since p, r analytic.

* If $p(z)$ has poles in D, $w(z)$ may acquire singularities.

Ex: $\frac{dw}{dz} - \nu \frac{w}{z} = 0$: $w = A z^{\nu}$

$\nu \neq$ integer: $z=0$ branch point ; ν integer >0 : analytic at $z=0$

ν integer <0 : pole at $z=0$

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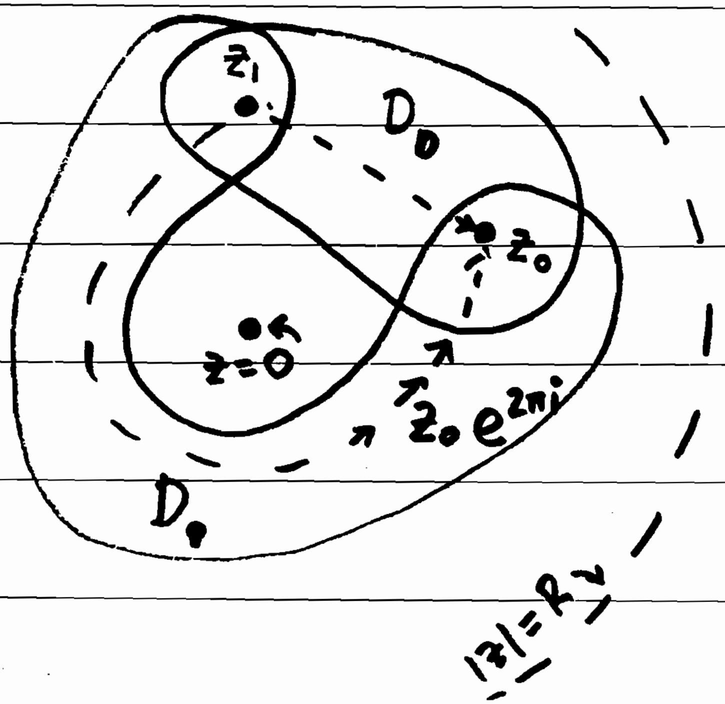
$$\frac{dw}{dz} + p(z)\,w = 0 \quad ; \; p(z) \text{ analytic on } 0<|z|<R$$

pole at $z=0$

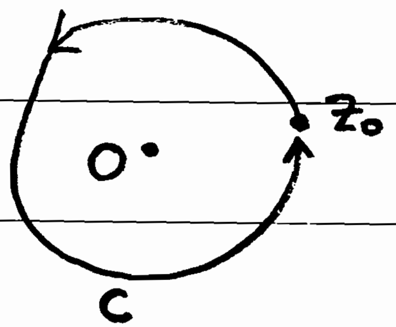


$w(z)$ analytic: any simply connected region D excluding $z=0$. Uniquely defined by $w(z_0)$, $z_0 \in D$. $w(z_0 e^{2\pi i})$ found by analytic continuation: let $z_1 \in D_0 \cap D_1$; $w(z_1)$ is uniquely defined from D_0. Now, $w(z_1)$ uniquely defines solution in D_1. This in turn gives $w(z_0 e^{2\pi i}) \neq w(z_0)$ in general. Then $w(z_0 e^{2\pi i}) = \lambda w(z_0)$

Exact solution: $w(z) = w(z_0)\, e^{P(z)}$

$$P(z) = \int_{z_0}^{z} p(\zeta)\, d\zeta$$

Find $w(z_0 e^{2\pi i}) = w(z_0) \underbrace{e^{\oint_c p(\zeta) d\zeta}}_{\equiv \lambda}$ (loop around $z=0$)



Let σ: $z^\sigma w(z)$ is single valued

$$(z e^{2\pi i})^\sigma w(z e^{2\pi i}) = z^\sigma e^{2\pi i \sigma} \lambda\, w(z) = z^\sigma w(z)$$

if $\lambda = e^{-2\pi i \sigma} \Rightarrow \sigma = -\frac{1}{2\pi i} \log \lambda$.

Since $z^\sigma w(z)$ single valued in $0<|z|<R$

has Laurent series $\Rightarrow$ $w(z) = z^{-\sigma} \sum_{-\infty}^{\infty} a_m z^m$

If $a_m = 0$, $m \leq -N$: $w(z) = z^{-\sigma-N}(b_0 + b_1 z + \cdots) = z^{-c} h(z)$ (analytic)

$\hookrightarrow$ regular singular point. (if and only if p has simple pole)

2^{nd} order, Constant Coeffs. $y = \kappa e^{\lambda x}$

$(*)\ y'' + ay' + by = 0 \longleftrightarrow \lambda^2 + a\lambda + b = 0$

(i) $\lambda_1 \neq \lambda_2$ real : $u_n = e^{\lambda_n x}$

(ii) $\lambda_1 = \lambda_2^* = p + iq$: $u = e^{px}\begin{cases}\cos qx \\ \sin qx\end{cases}$

(iii) λ (equal roots) : $e^{\lambda x}$, $xe^{\lambda x}$

$\left(e^{px} \frac{\sin qx}{q} \xrightarrow{q \to 0} xe^{px} \right).$

Linear/homogeneous: Superposition $u = c_1 u_1 + c_2 u_2$

Linear independence : $\alpha u_1(x) + \beta u_2(x) \overset{\forall x}{=} 0 \Rightarrow \alpha, \beta = 0$

$\Rightarrow \alpha u_1' + \beta u_2' = 0$

independent: $\begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} \neq 0$

In general, n functions $\phi_1, \ldots, \phi_n$ are independent:

$$\left.\begin{aligned} c_1\phi_1 + \cdots + c_n\phi_n &= 0 \\ c_1\phi_1' + \cdots + c_n\phi_n' &= 0 \\ &\vdots \\ c_1\phi_1^{(n-1)} + \cdots + c_n\phi_n^{(n-1)} &= 0 \end{aligned}\right\}$$

At any point x, this is a system of n-equations in the n unknowns $c_1, \ldots, c_n$.

is $\det\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix}$ $\nearrow = 0$: nontrivial solution (dependent)
$\searrow \neq 0$: only trivial solution (indep).

Let y solution of $*$; u_1 and u_2 lin. indep. solns of $*$:

$$\begin{vmatrix} u_1 & u_2 & y \\ u_1' & u_2' & y' \\ u_1'' & u_2'' & y'' \end{vmatrix} = 0$$

$u_1'' = -(au_1' + bu_1)$, $u_2'' = -(au_2' + bu_2)$, $y'' = -(ay' + by)$

(any 3 solns. of $*$ are dependent)

Wronskian $W(x) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$

$$\frac{dW}{dx} = \begin{vmatrix} u_1 & u_2 \\ u_1'' & u_2'' \end{vmatrix} = - \begin{vmatrix} u_1 & u_2 \\ au_1'+bu_1 & au_2'+bu_2 \end{vmatrix} = -aW$$

(This and the previous conclusion also valid if a, b are functions of x)

$$\Rightarrow W(x) = W(x_0)\, e^{-\int_{x_0}^{x} a(z)\,dz}$$

So either $W(x) \equiv 0$ (if $W(x_0)=0$), or $W(x) \neq 0$ (identically zero if it vanishes at one place - else nowhere zero).

Fundamental set: $\begin{matrix} u_1(x_0)=1 & u_2(x_0)=0 \\ u_1'(x_0)=0 & u_2'(x_0)=1 \end{matrix}$ $(W(x_0)=1)$

Now, let $y = k_1 u_1(x) + k_2 u_2(x)$

At $x = \alpha$:

$$y(\alpha) = k_1 u_1(\alpha) + k_2 u_2(\alpha)$$

$$y'(\alpha) = k_1 u_1'(\alpha) + k_2 u_2'(\alpha)$$

Now $\det\begin{pmatrix} u_1(\alpha) & u_2(\alpha) \\ u_1'(\alpha) & u_2'(\alpha) \end{pmatrix} = W(\alpha) \neq 0$ if

u_1, u_2 independent

$\Rightarrow$ can solve for k_1, k_2 given $y(\alpha), y'(\alpha)$.

$\Rightarrow$ Given fundamental set (or any two indep. solutions), any other solution is completely specified by its value and derivative at a point.

Inhomogeneous equ.

$y'' + ay' + by = r(x)$:

solution $v(x)$: particular

Then $y(x) = v(x) + c_1 u_1 + c_2 u_2$ general

(since difference of any two particular solutions satisfies homo. equ.).

Variation of parameters : let $v(x) = c_1(x) u_1(x) + c_2(x) u_2(x)$

$$v'(x) = \underbrace{c_1' u_1 + c_2' u_2}_{=0 \text{ by design}} + c_1 u_1' + c_2 u_2'$$

$$v'' = c_1' u_1' + c_2' u_2' + c_1 u_1'' + c_2 u_2''$$

plug in

$$\Rightarrow \begin{cases} c_1' u_1 + c_2' u_2 = 0 \\ c_1' u_1' + c_2' u_2' = r(x) \end{cases}$$

solve for c_1', c_2'

$\rightarrow$ integrate.