# Bessel's equation

$$w'' + \frac{w'}{z} + \left(1 - \frac{\nu^2}{z^2}\right) w = 0$$

solutions: BESSEL functions.

Origin: e.g., 2-dimensional Helmholtz equ.

$$\Delta\phi + k^2\phi = 0$$

Polar coords: $\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2} + k^2\phi = 0$

(even earlier: 2-d wave equation

$$\Box^2\phi = 0 \Rightarrow \Delta\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}$$

"Monochromatic solutions: $\Phi(x,y,t) = \phi(x,y)e^{i\omega t}$

Gives $\Delta\phi = -\frac{\omega^2}{c^2}\phi = -k^2\phi$, $k = \omega/c$. )

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Cylindrical geometries; solutions of form $\phi = f(r)\begin{cases}\cos n\theta \\ \sin n\theta\end{cases}$

$$\Rightarrow \frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(k^2 - \frac{n^2}{r^2}\right)f = 0$$

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Set $x = kr \ (= \frac{\omega}{c} r)$:

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} + \left(1 - \frac{n^2}{x^2}\right) f = 0$$

Go to complex variables $x \longrightarrow z$

Facts: $z = 0$ regular singularity
$z = \infty$ irregular "

Solutions about $z = 0$:

let $w = z^c (a_0 + a_1 z + \cdots)$:

$z^c$: $a_0^2 (c^2 - \nu^2) = 0 \quad \Rightarrow \quad c = \pm \nu$

$a_1 [(c+1)^2 - \nu^2] = 0 \qquad a_2/a_0 = - \frac{1}{2(2+2\nu)}$

$a_n = - \frac{a_{n-2}}{(c+n)^2 - \nu^2}, \ n \geq 2, \qquad a_4/a_0 = \frac{1}{2\cdot 4(2+2\nu)(4+2\nu)}$

$a_n = \frac{-a_{n-2}}{n(n+2\nu)} \qquad \frac{a_n}{a_0} = \frac{(-1)^k}{k!\, 2^{2k} (\nu+1)(\nu+2)\cdots(\nu+k)}$

Finally: Bessel fn. of $1^{st}$ kind, order $\nu$

$$J_\nu(z) = W =$$
$$= \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \left\{1 - \frac{1}{\nu+1}\left(\frac{z}{2}\right)^2 + \frac{1}{2!(\nu+1)(\nu+2)}\left(\frac{z}{2}\right)^4 - \cdots\right\}$$

normalization

solution exists if $\nu \neq$ negative integer.

$J_\nu(z)$, $J_{-\nu}(z)$ linearly independent.

Then $J_\nu(z)$ behaves like $z^\nu$ near $z=0$; it is finite

but $J_{-\nu}(z)$ behaves like $z^{-\nu}$ and is infinite at $z=0$.

e.g.: $$J_{1/2}(z) = \sqrt{\frac{2z}{\pi}}\,\frac{\sin z}{z}$$

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z$$

Problem if $\nu$ is negative integer.

The case ~~$\nu$~~ $\nu = n$, an integer:

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \left\{ 1 - \frac{1}{n+1}\left(\frac{z}{2}\right)^2 + \frac{1}{2!\,(n+1)(n+2)}\left(\frac{z}{2}\right)^4 + \dots \right\}$$

(homework 2)!

But can be shown that $J_\nu \to (-1)^n J_n$ as $\nu \to -n$: so $J_{-n}$ does not provide a second independent solution. To find a second solution, follow the general theory procedure for the case when the two roots of the indicial equ. differ by integer: let $u_1(z)$ be first series solution. Then try second:

$$u_2(z) = u_1(z) \log z + z^{-n} \sum_0^{\infty}{}' a_k z^k$$

Substitute this into Bessel equ. determine the $a_k$

Alternatively, let $w = v(z)\, J_n(z)$; then

$$v' = \frac{1}{z\, J_n^2(z)} \Rightarrow v(z) = \int_{z_0}^{z} \frac{ds}{s\, J_n^2(s)}.$$

Frobenius method ~~method~~) simpler:

let

$$Y_\nu(z) = \frac{1}{\sin \pi\nu} \left\{ J_\nu(z) \cos \pi\nu - J_{-\nu}(z) \right\}$$

$Y_\nu(z)$: Neumann fns. (Bessel of $2^{nd}$ kind).

If $\nu \neq n$, $Y_\nu$ gives a second soln., indep. of $J_\nu$.

What about $\nu \to n$? Define Bessel operator

$$B_n \equiv z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + (z^2 - n^2)$$

Clearly

$$B_n J_\nu = (\nu^2 - n^2) J_\nu, \quad B_n J_{-\nu} = (\nu^2 - n^2) J_{-\nu}$$

Hence: $B_n Y_\nu = \frac{\nu^2 - \pi^2}{\sin \pi\nu}\left(J_\nu \cos \pi\nu - J_{-\nu}\right)$

As $\nu \to n$, $J_\nu \to J_n$, $J_{-\nu} \to (-1)^n J_n$,

$\cos \pi\nu \to (-1)^n$

$\therefore\ B_n Y_\nu \to 0$ as $\nu \to n$. Since $Y_\nu$

analytic fn. of $\nu$,

$\Rightarrow B_n Y_n = 0$

with $\quad Y_n = \lim_{\nu \to n} \frac{1}{\sin \pi\nu}\left(J_\nu \cos \pi\nu - J_{-\nu}\right)$.

$\therefore\ Y_n(z)$ is desired second solution. Evaluate:

$$Y_n = \frac{1}{\pi}\left(\frac{\partial J_\nu}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}}{\partial \nu}\right)_{\nu = n}$$

Ex: $Y_0(z)$:

$$\left(\frac{\partial J_{-\nu}}{\partial \nu}\right)_{\nu=0} = -\left(\frac{\partial J_{-\nu}}{\partial(-\nu)}\right)_{\nu=0} = -\left(\frac{\partial J_\nu}{\partial \nu}\right)_{\nu=0}$$

$$\Rightarrow \quad Y_0(z) = \frac{2}{\pi}\left(\frac{\partial J_\nu}{\partial \nu}\right)_{\nu=0}$$

Since $J_\nu(z)$ has the expansion

$$J_\nu = \frac{e^{\nu \log(z/2)}}{\Gamma(\nu+1)}\left\{1 - \frac{1}{\nu+1}\left(\frac{z}{2}\right)^2 + \cdots\right\} \equiv \chi = \{\ldots\}$$

where: $\left(\frac{z}{2}\right)^\nu = e^{\nu \log(z/2)}$

$$\Rightarrow \frac{\partial J_\nu}{\partial \nu} = \left\{\frac{-\Gamma'(\nu+1)}{\Gamma^2(\nu+1)}\left(\frac{z}{2}\right)^\nu + \left(\log\frac{z}{2}\right)\left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}\right\}\left\{1 - \frac{1}{\nu+1}\left(\frac{z}{2}\right)^2 + \cdots\right\}$$

$$+ \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \left\{-\left(\frac{z}{2}\right)^2 \frac{\partial}{\partial \nu}\frac{1}{\nu+1} + \left(\frac{z}{2}\right)^4 \frac{1}{2!}\frac{\partial}{\partial \nu}\frac{1}{(\nu+1)(\nu+2)} + \cdots\right\}$$

Finally:

$$\left(\frac{\partial J_\nu}{\partial \nu}\right)_{\nu=0} = \left\{\log \frac{z}{2} - \Gamma'(1)\right\} J_0(z) +$$

$$+ \left\{\left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^4 \frac{1}{(2!)^2}\,\frac{3}{2} + \cdots\right\}$$

<u>Aside</u>: What is $\Gamma'(\nu+1)$ ?

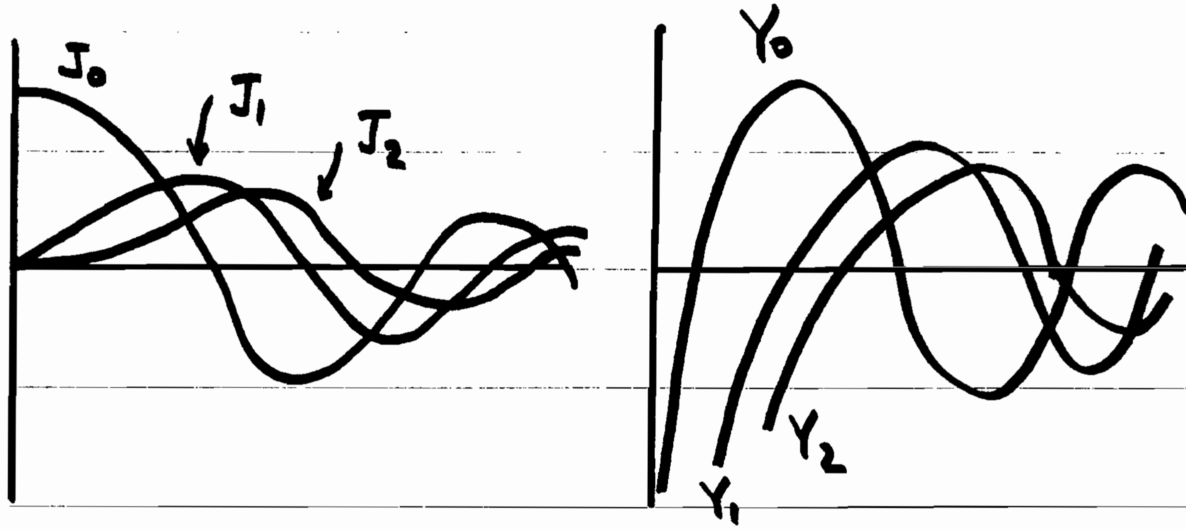
$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu \, dt,$$

$$\Gamma'(1) = \int_0^\infty e^{-t} \log t \, dt = -\gamma \simeq -.5772 \quad \text{(Euler's constant).}$$

$$\Rightarrow Y_0(z) = \frac{2}{\pi}\left\{\log \frac{z}{2} + \gamma\right\} J_0(z) + \frac{2}{\pi}\,\frac{z^2}{4}\left\{1 - \left(\frac{z}{2}\right)^2 \frac{1}{(2!)^2}\,\frac{3}{2} + \cdots\right\}$$

Similarly, can calculate all the $Y_n(z)$. Near $z=0$:

$$Y_n(z) = -\frac{2^n}{\pi\,(\cancel{2}n-1)!}\,\frac{1}{z^n} + \text{smaller terms.}$$




## GENERATING FUNCTION:

$$F(z;t) \quad e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{-\infty}^{\infty} J_n(z)\, t^n \qquad (\circledast)$$

RHS: Laurent expansion of $F(z,t)$ w.r.t. $t$. Show:

$$(\dagger) \qquad J_n(z) = \frac{1}{2\pi i}\oint \frac{e^{\frac{z}{2}(t-1/t)}}{t^{n+1}}\, dt$$

↳ simple close curve around 0.

let $u = \frac{1}{2} z t$:

$$\frac{1}{2\pi i}\oint \frac{\frac{z}{2}(t-\frac{1}{t})}{t^{n+1}}\, dt = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^n \oint \frac{e^u e^{-z^2/4u}}{u^{n+1}}\, du \qquad (*)$$

Expand $e^{-z^2/4u}$ in powers of $z^2/4u$:

$$e^{-z^2/4u} = 1 - \frac{z^2}{4u} + \dots + \frac{(-1)^k}{k!}\left(\frac{z^2}{4u}\right)^k + \dots$$

Plug into $(*)$:

$$(*) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{z}{2}\right)^{2k} \underbrace{\frac{1}{2\pi i}\oint \frac{e^u}{u^{n+k+1}}\,du}_{= \frac{1}{(n+k)!}}$$

$\Rightarrow$ (residue theorem)

So $$(*) = \dots = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{(n+k)!}\left(\frac{z}{2}\right)^{2k} = J_n(z)$$

**Properties**

In $(\dagger)$ let $t = e^{i\vartheta}$:

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\vartheta} e^{iz\sin\vartheta}\,d\vartheta =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(n\vartheta - z\sin\vartheta)\,d\vartheta - \frac{i}{2\pi}\int_0^{2\pi} \sin(n\vartheta - z\sin\vartheta)\,d\vartheta$$

Since $\sin(n\theta - z\sin\theta)$ is periodic, change limits of integration from $0 \to 2\pi$ to $-\pi \to \pi$. But
$$\int_{-\pi}^{\pi} \sin(n\theta - z\sin\theta)\, d\theta = 0$$
(odd integrand).

$$\Rightarrow \quad J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - z\sin\theta)\, d\theta$$

**INTEGRAL REPRESENTATION** of $J_n(z)$

Note: $|\cos(n\theta - x\sin\theta)| < 1$, $x, \theta$ real $\Rightarrow |J_n(x)| < 1$.

**RECURRENCE RELATIONS:**

$$\circledast \Rightarrow \frac{d}{dz}(\otimes) : \quad \frac{1}{2}\left(t - \frac{1}{t}\right)\sum_{n=-\infty}^{\infty} J_n(z)\, t^n = \sum_{n=-\infty}^{\infty} J_n'(z)\, t^n$$

or $$\sum{}' \frac{1}{2}\left(J_{n-1}(z) - J_{n+1}(z)\right) t^n = \sum J_n'(z)\, t^n$$

$$\Rightarrow \quad 2J_n'(z) = J_{n-1} - J_{n+1} \qquad (1)$$

$$\frac{d}{dt}(\otimes): \quad \frac{z}{2}\left(1+\frac{1}{t^2}\right)\sum J_n t^n = \sum n J_n t^{n-1}$$

$$\Rightarrow \frac{z}{2}\sum\left(J_{n-1}+J_{n+1}\right)t^{n-1} = \cancel{\phantom{n}} - \sum n J_n t^{n-1}$$

$$\Rightarrow \frac{2n}{z} J_n = J_{n-1} + J_{n+1} \qquad (2)$$

Also

$(2)-(1) \Rightarrow 2J_{n+1} = \frac{2n}{z} J_n - 2J_n' \Rightarrow \quad J_{n+1} = \frac{n}{z} J_n - J_n' \quad (3).$

Ex: $J_1 = -J_0'$ (cf. the graph)

But $J_n = (-1)^n J_{-n}$ ; with

[similar identities hold for the $Y_n$ trickier proofs.]

$(3) \Rightarrow \frac{d}{dz}\left(z^{-n}J_n\right) = -z^{-n}J_{n+1}$ and if we set $k+1 = -n$ :

$$\frac{d}{dz}\left(z^{k+1}J_{k+1}(z)\right) = z^{k+1}J_k(z) \quad \forall k \qquad (4)$$

# ASYMPTOTIC BEHAVIOR $x \to \infty$

let $y = \sqrt{x}\, J_n(x)$ :

$$J_n'' + \frac{1}{x} J_n' + \left(1 - \frac{n^2}{x^2}\right) J_n = 0$$

$$\Rightarrow \quad y'' + \left(1 - \frac{n^2 - 1/4}{x^2}\right) y = 0$$

$x = \infty$ is irregular singularity: no power series.

But, as $(n^2 - 1/4)/x^2 \xrightarrow{x \to \infty} 0$, try $y \sim \cos x$ or $\sin x$

i.e. let $y(x) = \cos x \,(a_0 + a_1/x + \dots) + \sin x\,(b_0 + b_1/x + \dots)$. (5)

Work for $n = 0$ : substitute (5) into $y'' + \left(1 + \frac{1}{4x^2}\right) y$ :

$$
\begin{aligned}
y'' + y \qquad & 2 \sin x \left(\frac{a_1}{x^2} + \frac{2a_2}{x^3} + \dots\right) + \cos x \left(\frac{2a_1}{x^3} + \dots\right) \\
& - 2\cos x \left(\frac{b_1}{x^2} + \frac{2b_2}{x^3} + \dots\right) + \sin x \left(\frac{2b_1}{x^3} + \dots\right) \\
\frac{1}{4x^2} y \qquad & + \cos x \left(\frac{a_0}{4x^2} + \frac{a_1}{4x^3} + \dots\right) + \sin x \left(\frac{b_0}{4x^2} + \frac{b_1}{4x^3} + \dots\right) = 0
\end{aligned}
$$

($a_0, b_0$ arbitrary)

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Equate coeffts. of $\frac{\cos x}{x^n}$, $\frac{\sin x}{x^n}$ to zero:

$n=0$: $-2b_1 + \frac{a_0}{4} = 0$, $2a_1 + \frac{b_0}{4} = 0$

$n=3$: $-4b_2 + 2a_1 + \frac{a_1}{4} = 0$, $4a_2 + 2b_1 + \frac{b_1}{4} = 0$

. . .

Given $a_0$ find $b_1, a_2, b_3, a_4, \ldots$

" $b_0$ find $a_1, b_2, a_3, b_4, \ldots$

(1) let $a_0 = 1, b_0 = 0$: $y_1 = \cos x + \frac{1}{8x}\sin x + \cdots$

(2) let $a_0 = 0, b_0 = 1$: $y_2 = \sin x - \frac{1}{8x}\cos x + \cdots$

But these series can be shown to diverge! Still, taking a finite number of terms provides valuable info on solutions!

$J_0(x)$ is related to $y_1(x), y_2(x)$:

$$J_0(x) = \frac{1}{\sqrt{x}}\left(A y_1(x) + B y_2(x)\right) \quad (6)$$

Find $A$, $B$ from integral representation:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\sin\theta)\,d\theta$$

let $\sin\theta = 1 - \frac{u^2}{x}$ ; then

$$J_0(x) = \frac{2}{\pi}\int_0^{\sqrt{x}}(\cos x\cos u^2 + \sin x\sin u^2)\sqrt{\frac{2}{x}}\underbrace{\left(1-\frac{u^2}{2x}\right)^{-1/2}}_{\sim 1}du$$

$$\sim \frac{2^{3/2}}{\pi\sqrt{x}}\int_0^{\infty}(\cos x\cos u^2 + \sin x\sin u^2)\,du \quad , \quad x \to \infty$$

But $\int_0^{\infty}\cos u^2\,du = \int_0^{\infty}\sin u^2\,du = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

$$\Rightarrow \quad J_0(x) \sim \frac{\cos x + \sin x}{\sqrt{\pi x}} = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right) \text{ as } x \to \infty.$$

So $A = B = 1/\sqrt{\pi}$ in (6)

Use recurrence formula

$$J_{n+1} = -J_n' + \frac{n}{x} J_n$$

to find asymptotic behavior of $J_n$, $n > 1$.

Find $$\boxed{J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \quad (7)$$

Result more general than derivation indicates: valid for $n \to \nu$, noninteger, $x \to z$, complex (but with $|\arg z| < \pi - \delta$, $\delta > 0$). Now can use Froberius definition to find asymptotic formulas for $Y_n(z)$:

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (8)$$

$$|\arg z| < \pi - \delta, \ \delta > 0.$$

<u>Hankel functions</u> : like in solving $y''+y=0$
$(\sin x, \cos x)$ or $(e^{ix}, e^{-ix})$: related
$e^{ix} = \cos x + i\sin x$ , $e^{-ix} = \cos x - i\sin x$
Similarly: (Hankel)

$$H_\nu^{(1)}(z) = J_\nu(z) + i\,Y_\nu(z)$$

$$H_\nu^{(2)}(z) = J_\nu(z) - i\,Y_\nu(z)$$

As $x \to \infty$:

$$H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}$$

$$H_\nu^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}$$

The Hankel functions are a second linearly independent pair of solutions of Bessel's equation. They satisfy same recurrence rels. as $J_\nu$, $Y_\nu$. Main application is in wave propagation problems with cylindrical symmetry.

# EIGENVALUE PROBLEM & ORTHOGONALITY

$$y'' + \frac{1}{x}y' + \left(k^2 - \frac{n^2}{x^2}\right)y = 0, \quad 0 \le y \le a \qquad (*)$$

$$y(a) = 0, \quad y(0) < \infty$$

$$\Rightarrow \quad y = A J_n(kx) + B Y_n(kx) \quad \underline{\text{general}}$$

$y(0) < \infty \Rightarrow B = 0$ i.e. $\underline{y(x) = A J_n(kx)}$

$y(a) = 0 \Rightarrow J_n(ka) = 0$ ; $ka = j_1, j_2, \ldots$; roots of $J_n$.

$\therefore$ There is a countable infinity of solutions, for $k_n = j_n/a$.

Let $y_r(x) = J_n(j_r x)$, $y_s(x) = J_n(j_s x)$. Write $(*)$ in self-adjoint form:

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(k^2 x - \frac{n^2}{x}\right)y = 0 \; :$$

$y_r$ satisfies: $y_s\left(\frac{d}{dx}\left(x\frac{dy_r}{dx}\right) + \left(k_r^2 x - \frac{n^2}{x}\right)y_r = 0\right)$ $\quad k_r = j_r/a$

similarly $\quad y_r\left(\frac{d}{dx}\left(x\frac{dy_s}{dx}\right) + \left(k_s^2 x - \frac{n^2}{x}\right)y_s = 0\right)$ $\quad k_s = j_s/a$

$$\Rightarrow \quad y_r\frac{d}{dx}\left(x\frac{dy_s}{dx}\right) - y_s\frac{d}{dx}\left(x\frac{dy_r}{dx}\right) = (k_r^2 - k_s^2)\,x\,y_r y_s$$

Integrate: $\int_0^a \left[y_r \frac{d}{dx}(\ldots) - y_s \frac{d}{dx}(\ldots)\right] dx =$

(since $\frac{d}{dx}\left(y_r x \frac{dy_s}{dx} - y_s x \frac{dy_r}{dx}\right) - \left[\cancel{x \frac{dy_r}{dx}\frac{dy_s}{dx} - x \frac{dy_s}{dx}\frac{dy_r}{dx}}\right]$)

$= \left(y_r x y_s' - y_s x y_r'\right)_0^a = (k_r^2 - k_s^2)\int_0^a x y_r y_s \, dx$

Since $y_r(a) = y_s(a) = 0 \Rightarrow$, if $k_r \neq k_s$,

$$\int_0^a x y_r y_s \, dx = 0.$$

i.e. $\int_0^a x J_n(k_r x) J_n(k_s x)\, dx = 0$; $k_r = \mathcal{J}_r/a$, $k_s = \mathcal{J}_s/a$, $\mathcal{J}_r \neq \mathcal{J}_s$

$\mathcal{J}_r, \mathcal{J}_s$ zeroes of $J_n(\mathcal{J}) = 0$.

## Modified Bessel functions

Define $I_\nu(x) = e^{-\frac{1}{2}\nu\pi i} J_\nu(ix) =$

$$= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \left\{ 1 + \frac{1}{\nu+1}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(\nu+1)(\nu+2)}\left(\frac{x}{2}\right)^4 + \dots \right\}$$

(mod. B.-functions or B-fns. of complex argument)

Note: $I_\nu(x) > 0$ for $x > 0$. The $I_\nu(x)$ satisfy:

$$z^2 w'' + z w' - (z^2 + \nu^2) w = 0. \qquad (\dagger)$$

If $\nu$ not $= n$, integer:

"ix"

Second solution of $(\dagger)$: $I_{-\nu}(x) = e^{\frac{1}{2}\nu\pi i} J_{-\nu}(ix)$

If $\nu = n$, $I_n = I_{-n}$. In this case, second solution $K_n(x)$:

$$K_\nu(x) = \lim_{\nu \to n} \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}$$

Near $x=0$:

$$I_\nu(x) \sim \left(\frac{x}{2}\right)^\nu, \quad K_\nu(x) \sim \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu}, \quad \nu>0$$

while $K_0(x) \sim -\left\{\log \frac{x}{2} + \gamma\right\} I_0(x)$

The behavior as $x \to \infty$ follows from the asymptotic formulas (7) & (8) for the $J_n$ & $Y_n$

$$I_\nu(x) = e^{-\frac{1}{2}\nu\pi i} J_\nu(ix) \sim e^{-\frac{1}{2}\nu\pi i}\sqrt{\frac{2}{\pi i x}} \cos\left(ix - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

since $\cos\left(ix - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = \left(e^{-x} e^{-\frac{i\nu\pi}{2}} e^{-i\frac{\pi}{4}} + e^{x} e^{\frac{i\nu\pi}{2}} e^{i\pi/4}\right)/2$

$$= \frac{\sqrt{i}}{2} e^{\pi\nu i/2} e^{x} + \frac{1}{2\sqrt{i}} e^{-\pi\nu i/2} e^{-x}$$

So: $$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^{x} + \frac{1}{i}\frac{1}{\sqrt{2\pi x}} e^{-i\pi\nu} e^{-x} \qquad (9)$$

From this, get for $K_\nu$: $$K_\nu = \frac{\pi}{2}\frac{I_{-\nu} - I_\nu}{\sin\nu\pi} \sim \sqrt{\frac{\pi}{2x}}\, e^{-x} \qquad (10)$$