FOURIER TRANSFORMS

Consider $f(x)$, $-\infty < x < \infty$, $\int_{-\infty}^{\infty} |f(\xi)|\, d\xi < \infty$.

Then $\int_{-\infty}^{\infty} f(\xi) e^{-ik(\xi - x)} dx$ is defined

(and $\lim_{L\to\infty} \int_{-L}^{L}$ converges uniformly)

Then $\frac{1}{2\pi} \int_{-A}^{A} \left(\int_{-\infty}^{\infty} f(\xi) e^{-ik(\xi - x)} d\xi \right) dk =$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \left(\int_{-A}^{A} e^{-ik(\xi - x)} dk \right) d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{\sin[A(\xi - x)]}{\xi - x} d\xi \equiv f_A(x)$$

Now $\int_0^{\infty} \frac{\sin Ax}{x} dx = \frac{\pi}{2}$ (done by residue thm.)

$$\left.\begin{aligned} \Rightarrow \tfrac{1}{2} f(x+0) &= \frac{1}{\pi} \int_{x}^{\infty} f(x+) \frac{\sin\{A(\xi - x)\}}{\xi - x} dx \\ \tfrac{1}{2} f(x-0) &= \frac{1}{\pi} \int_{-\infty}^{x} f(x-) \frac{\sin\{A(\xi - x)\}}{\xi - x} dx \end{aligned}\right\} \quad (*)$$

$$f_A(x) - \frac{1}{2}\left(f(x^+) + f(x^-)\right) =$$

$$\left.\begin{aligned} &\frac{1}{\pi}\int_x^{\infty} \frac{f(\xi) - f(x^+)}{\xi - x} \sin A(\xi - x)\, d\xi \\ &+ \frac{1}{\pi}\int_{-\infty}^{x} \frac{f(\xi) - f(x^-)}{\xi - x} \sin A(\xi - x)\, d\xi \end{aligned}\right\} (*)$$

If $f(\xi)$ has right & left-hand derivatives at $\xi = x$

then $g_1 = \dfrac{f(\xi) - f(x^+)}{\xi - x}$, $g_2 = \dfrac{f(\xi) - f(x^-)}{\xi - x}$

are continuous in $\xi \geq x$ and $\xi \leq x$ resp. Then $(*)$ goes to zero by the Riemann-Lebesgue lemma:

if $f \in L^1(a,b)$ (i.e. $\int_a^b |f|\, dx < \infty$) then $\lim_{k\to\infty} \int_a^b f(x) e^{ikx}\, dx = 0$

(~~Proof for $g \in L^2(a,b)$~~

~~$|\int g e^{ikx} dx|$~~

So: let $F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$

$(f(x) \in L^1(\mathbb{R}))$; defined for k real

$\Rightarrow$ $f(x) = \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$

Fourier transform pair

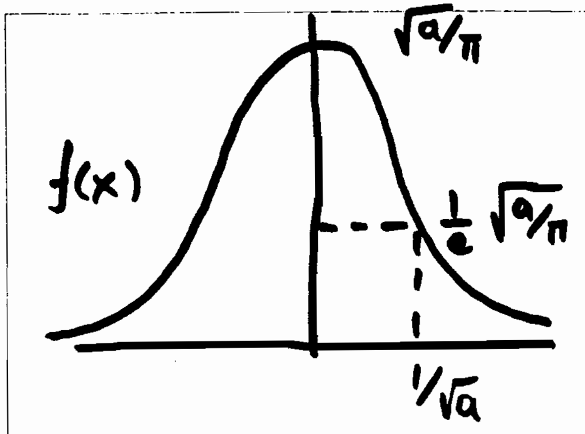
Ex: $f(x) = N e^{-ax^2}$

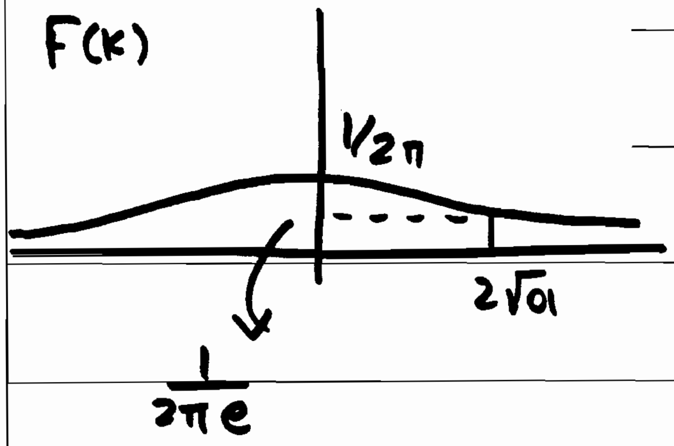
$$F(k) = \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-ax^2 - ikx}\, dx =$$

$$= \frac{N}{2\pi} e^{-k^2/4a} \int_{-\infty}^{\infty} e^{-a\left(x + \frac{ik}{2a}\right)^2} dx = \frac{1}{2}\frac{N}{\sqrt{\pi a}} e^{-k^2/4a}$$

If we require $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow N = \sqrt{\frac{a}{\pi}}$ find

$$f(x) = \sqrt{\frac{a}{\pi}}\, e^{-ax^2}, \quad F(k) = \frac{1}{2\pi} e^{-k^2/4a}$$





Now: $f(x) \to \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$ as $a \to \infty$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \forall a.$$

Dirac delta function:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

Now $F(k) \to \frac{1}{2\pi}$ as $a \to \infty$. So

$$\left.\begin{aligned} \frac{1}{2\pi} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\delta(x)\,dx \\ \delta(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx \end{aligned}\right\}$$

(classically divergent!)

Properties F.T.

let $\hat{y}(k) = \mathcal{F}\{y(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(x)\,e^{-ikx}\,dx$

$$\mathcal{F}\left\{\frac{dy}{dx}\right\} = ik\,\hat{y}(k)$$

$$\mathcal{F}\left\{\frac{d^2y}{dx^2}\right\} = -k^2\,\hat{y}(k)$$

Convolution: $f * g = \int_{-\infty}^{\infty} f(\xi)\,g(x-\xi)\,d\xi$; $\mathcal{F}\{f*g\} = 2\pi\hat{f}\,\hat{g}$

Convolution

$\mathcal{F}\{f*g\} = F\,G$

$F = \hat{f},\ G = \hat{g}$:

$$\int_{-\infty}^{\infty} F(k)\,G(k)\,e^{ikx}\,dk$$

$$\parallel$$

$$f*g\,(x)$$

$$\parallel$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,g(x-\xi)\,d\xi \quad (*)$$

Parseval's theorem: set $x=0$ in convolution $(*)$

$$\int_{-\infty}^{\infty} F(k)G(k)\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,g(-\xi)\,d\xi$$

But, if $g(-\xi) = h(\xi) \Rightarrow H(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(\xi)\,e^{-ik\xi}\,d\xi$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(-\xi)\,e^{-ik\xi}\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\,e^{ik\xi}\,d\xi = \bar{G}(k)$$

So we set $g(-\xi) = \bar{f}(\xi)$, and compute: $\left(g(\xi) = \bar{f}(-\xi)\right)$

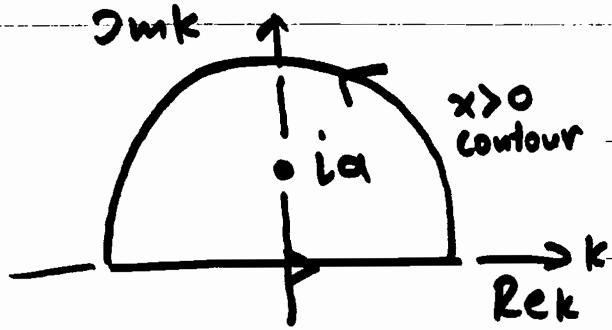
$$G(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,\bar{f}(-x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\bar{f}(x)\,dx = \bar{F}(k)$$

$$\therefore \int_{-\infty}^{\infty} |F|^2\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f|^2\,dx$$

(Proof of convolution:

$$\int_{-\infty}^{\infty} F\,G\,e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} G\,e^{ikx}\left(\int_{-\infty}^{\infty} f(\xi)\,e^{-ik\xi}\,d\xi\right)dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, f(\xi)\left(\int_{-\infty}^{\infty} G(k)\,e^{ik(x-\xi)}\,dk\right)$$

19.6

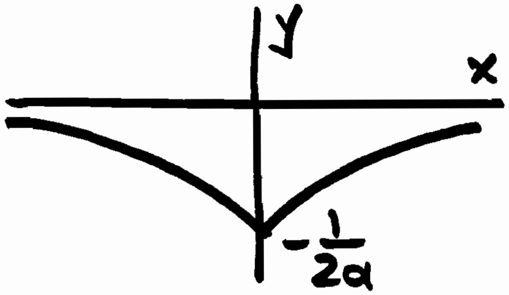


Ex: $\frac{d^2y}{dx^2} - a^2 y = g(x) \quad , \quad -\infty < x < \infty$

$y(\infty) = y(-\infty) = 0$

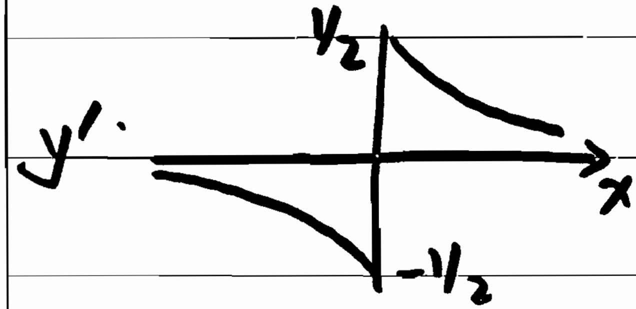
$\Rightarrow -(k^2+a^2)Y = G \; : \; Y = -\frac{G(k)}{k^2+a^2}$

$y(x) = -\int_{-\infty}^{\infty} \frac{G(k)}{k^2+a^2} e^{ikx}\, dk.$



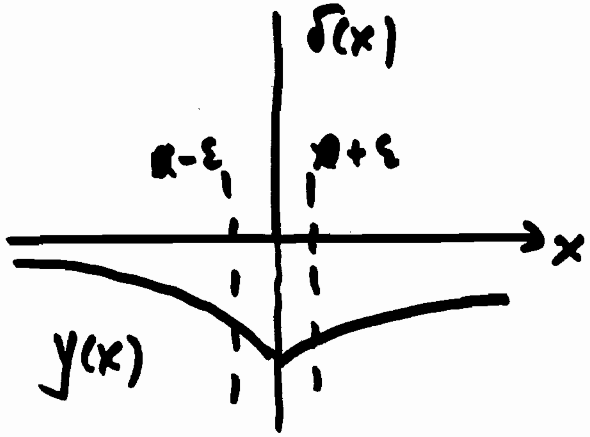
Let $g(x) = \delta(x)$; $G(k) = \frac{1}{2\pi}$

$$y(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+a^2}\, dx = \begin{cases} -\frac{1}{2a} e^{-ax} , & x>0 \\ -\frac{1}{2a} e^{ax} , & x<0 \end{cases} = -\frac{1}{2a} e^{-a|x|}$$



Derivative discontinuous.

19.2



Integrate $\displaystyle\int_{-\varepsilon}^{+\varepsilon} \left(\frac{d^2y}{dx^2} - a^2 y = \delta(x) \right) dx$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{-\varepsilon}^{+\varepsilon} - a^2 \int_{-\varepsilon}^{\varepsilon} y(x)\,dx = 1$$

$\searrow 0$, y continuous

$$\Rightarrow \left[\frac{dy}{dx}\right]_{-\varepsilon}^{\varepsilon} \rightarrow 1 \text{ as } \varepsilon \rightarrow 0.$$

i.e. derivative jumps by 1 at $x=0$, while y continuous.

<u>Alternative approach</u>: solve $y'' - a^2 y = \delta(x)$, $y(\pm\infty) = 0$

$x<0 \quad y'' - a^2 y = 0 \quad \Rightarrow y = A_- e^{ax} + \underbrace{\cancel{B_-}}_{=0} e^{-ax}$

$x>0 \quad y'' - a^2 y = 0 \quad\quad y = \underbrace{\cancel{A_+}}_{=0} e^{ax} + B_+ e^{-ax}$

$A_+ = B_- = 0$ since $y(\pm\infty) = 0$.

Continuity $x=0$: $A_- = B_+$

Jump $y'|_{-\varepsilon}^{\varepsilon} \rightarrow 1 \Rightarrow -aB_+ - aA_- = 1 \Rightarrow B_+ = A_- = \dfrac{-1}{2a}$

Generalization : $\frac{d^2y}{dx^2} - a^2 y = \delta(x-\xi)$

solution $G(x|\xi)$ **Green's function**

let $x^* = x - \xi$: $\frac{d^2y}{dx^{*2}} - a^2 y = \delta(x^*)$;

$y = -\frac{1}{2a} e^{-k|x^*|}$

$$G(x|\xi) = -\frac{1}{2a} e^{-k|x-\xi|}$$

Inhomogeneous problem: $\frac{d^2y}{dx^2} - a^2 y = g(x)$, $y(\pm\infty) = 0$.

Convolution: $g(x) * \delta(x) = \int_{-\infty}^{\infty} g(\xi)\,\delta(x-\xi)\,d\xi \xrightarrow{\mathcal{F}} 2\pi \cdot \frac{1}{2\pi} G(k) = G(k)$

i.e. $g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)\,d\xi$

Define $L = \frac{d^2}{dx^2} - a^2$; then $L\,G(x|\xi) = \delta(x-\xi)$

$\Rightarrow \int_{-\infty}^{\infty} g(\xi)\, L\, G(x|\xi)\, d\xi = \int_{-\infty}^{\infty} g(\xi)\,\delta(x-\xi)\,d\xi = g(x)$

L indep. of x $\longrightarrow$ solution

$\Rightarrow L\left\{ \int_{-\infty}^{\infty} g(\xi)\, G(x|\xi)\, d\xi \right\} = g(x)$

Fourier sine & cosine transforms

(i) $f(x) = f(-x)$: even

$$F(k) = \frac{1}{2\pi}\left(\int_0^\infty + \int_{-\infty}^0\right) f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\pi}\int_0^\infty f(x)\cos kx\, dx \equiv F_c(k)$$

(ii) $F_c(k)$ even

$$f(x) = 2\int_0^\infty F_c(k)\cos kx\, dk$$

Similarly $f(x) = -f(-x)$ odd

$$\left.\begin{aligned} F_s(k) &= \frac{1}{\pi}\int_0^\infty f(x)\sin kx\, dx \\ f(x) &= 2\int_0^\infty F_s(k)\sin kx\, dx \end{aligned}\right\}$$

($F_s(k) = -F_s(-k)$, odd also).

Ex: $y'' + a^2 y = \delta(x-\xi)$, $y(\pm\infty) < \infty$

$$y = \begin{cases} A_+ e^{ia(x-\xi)} + B_+ e^{-ia(x-\xi)}, & x > \xi \\ A_- e^{ia(x-\xi)} + B_- e^{-ia(x-\xi)}, & x < \xi \end{cases}$$

Boundedness automatically satisfied at $\pm\infty$.

Continuity: $A_+ + B_+ = A_- + B_-$ $\quad (x = \xi)$

Jump: $\left[\dfrac{dy}{dx}\right]_{\xi^-}^{\xi^+} = 1$: $ia(A_+ - B_+) - ia(A_- - B_-) = 1$

$\therefore$ only two relations for A_+, B_+, A_-, B_- !

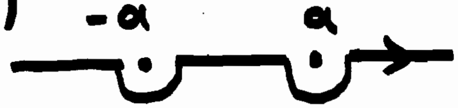
Additional conditions:

(1) **Causality:** $y(-\infty) = 0$, $A_- = B_- = 0$: $\cancel{A_+ = B}$

$$\Rightarrow A_+ = -B_+ = \frac{1}{2ia}$$

$$y(x) = \begin{cases} \frac{1}{a} \sin a(x-\xi), & x > \xi \\ 0 & x < \xi \end{cases}$$

(1)



(2) Radiation:

$$y \to \begin{cases} k e^{iax} & x \to \infty \\ k e^{-iax}, & x \to -\infty \end{cases}$$

$$A_- = B_+ = 0, \quad A_+ = B_- = \frac{1}{2ia}$$

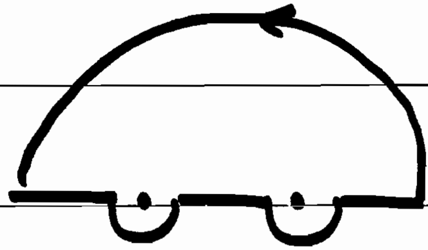
(2)

$$y = \begin{cases} \frac{1}{2ia} e^{ia(x-\xi)}, & x > \xi \\ \frac{1}{2ia} e^{-ia(x-\xi)}, & x < \xi \end{cases}$$

Getting same result by Fourier x-form

$$(a^2-k^2)Y = \frac{1}{2\pi} \Rightarrow Y = \frac{1}{2\pi(a^2-k^2)} \Rightarrow y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2-k^2}\,dk$$

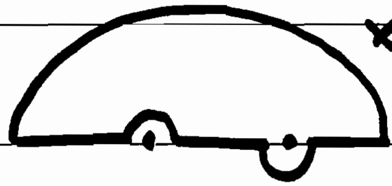
Path (1)



$x>0$

both poles

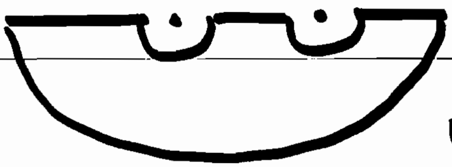
causality contour



$x>0$

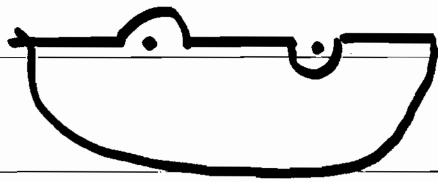
radiation contour

$x<0$

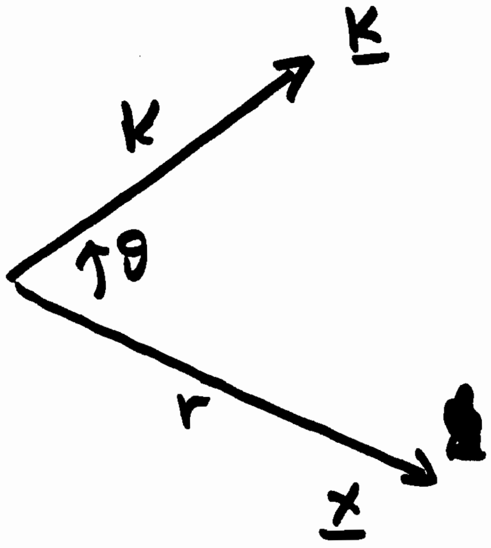


$x<0$

no poles



$x<0$



2-d Fourier xforms

$$F(k,l) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} f(x,y)\, e^{-i(kx+ly)}\, dx\,dy \qquad (1)$$

$$\left(\text{need } \iint_{-\infty}^{\infty} |f(x,y)|\, dx\,dy < \infty\right)$$

Inversion: $$f(x,y) = \iint_{-\infty}^{\infty} F(k,l)\, e^{i(kx+ly)}\, dk\,dl \qquad (2)$$

Special forms: $f(x,y) = h(r)$, $r^2 = x^2+y^2$ (radial symmetry)

Use polars: $\underline{x} = x\underline{i} + y\underline{j}$, $\underline{K} = k\underline{i} + l\underline{j}$. From figure

$K = \sqrt{k^2+l^2}$, $r = \sqrt{x^2+y^2}$, $kx + ly = \underline{K}\cdot\underline{x} = Kr\cos\theta$,

$dx\,dy = r\,dr\,d\theta$.

$$H(K) = \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{2\pi} h(r)\, e^{-iKr\cos\theta}\, r\,d\theta\,dr = \qquad (1)'$$

$$= \frac{1}{4\pi^2}\int_0^{\infty} r\,h(r) \left\{\int_0^{2\pi} e^{-iKr\cos\theta}\, d\theta\right\} dr$$

$$h(r) = \int_0^\infty k H(k) \left\{ \int_0^{2\pi} e^{ikr\cos\vartheta} d\vartheta \right\} dk$$

Now $\displaystyle\int_0^{2\pi} e^{ikr\cos\vartheta} d\vartheta = 2\pi J_0(kr)$:

$$\left.\begin{aligned} H(k) &= \frac{1}{2\pi}\int_0^\infty r\, h(r)\, J_0(kr)\, dr \\ h(r) &= 2\pi \int_0^\infty k H(k)\, J_0(kr)\, dk \end{aligned}\right\} *$$

<u>Hankel transform pair</u> ((*) true if we replace J_0 by J_ν as well).