Contour Integrals

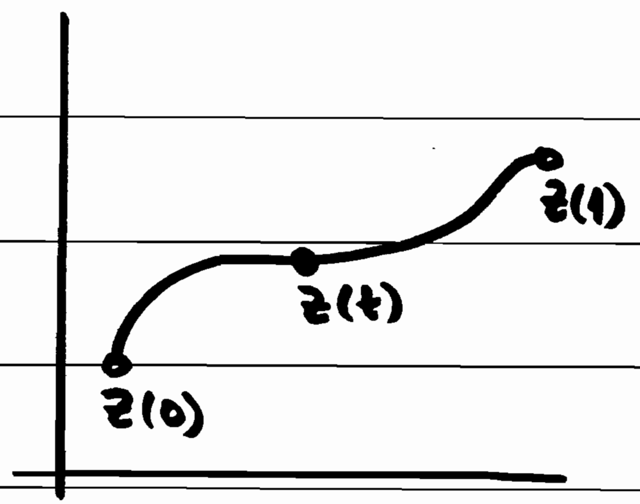
$$\int_{z_0\,C}^{z_1} f(z)\,dz = \int_{(x_0,y_0)}^{(x_1,y_1)} (u+iv)(dx+i\,dy)$$

Parametrized curve $(x(t), y(t))$.

Ex: $$\oint_{|z|=1} z^\alpha\, dz \underset{z=e^{i\theta}}{=} \int_{\theta=0}^{2\pi} e^{i\alpha\theta}\, i e^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} e^{i(\alpha+1)\theta}\, d\theta =$$

$$\begin{cases} \alpha=-1: & = 2\pi i \\ \alpha\neq -1,\ \text{integer}: & = \frac{1}{\alpha+1}\left[e^{i(\alpha+1)2\pi}-1\right] = 0 \\ \alpha \text{ non-integer}: & = \frac{1}{\alpha+1}\left[e^{i(\alpha+1)2\pi}-1\right] \neq 0 \end{cases}$$



Path: $(x(t), y(t)) \sim z(t)$

$dz = (\dot{x} + i\dot{y})\,dt$

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Ex: $f(z) = F'(z)$; $F = U + iV$

$$\int f(z)\,dz = \int_c (U_x\,dx + U_y\,dy) + i\int_c (V_x\,dx + V_y\,dy)$$

$$= \int_c \nabla U \cdot d\underline{r} + i\int_c \nabla V \cdot d\underline{r}$$

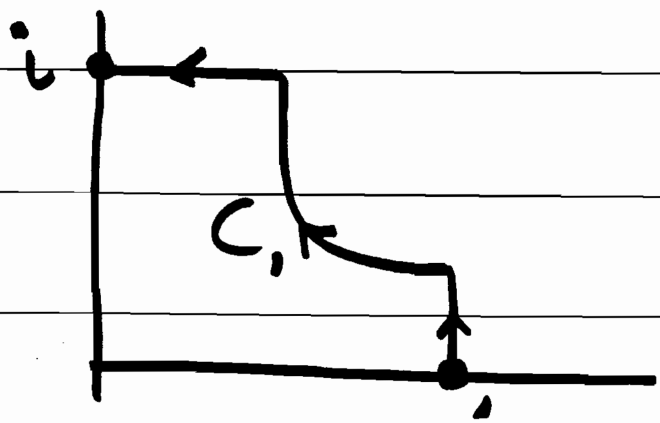
(real line integrals)

$$\Rightarrow \int_c f(z)\,dz = U(z_1) - U(z_0) + i\left(V(z_1) - V(z_0)\right)$$

$$= F(z_1) - F(z_0)$$

$$\left.\begin{aligned} u &= U_x = V_y \\ v &= V_x = -U_y \end{aligned}\right\}$$

Ex: $\displaystyle\int_{C_1} z\,dz = \frac{1}{2} z^2 \Big|_{z=1}^{z=i} = -\frac{1}{2} - \frac{1}{2} = -1$



Cauchy Integral Theorem

if $\mathcal{R}$ is simply connected region where $f(z)$ is analytic, then $\oint_c f(z)dz = 0$ for any closed contour C in $\mathcal{R}$.

$$\oint f dz = \oint \underset{①}{(udx - vdy)} + i\oint \underset{②}{(vdx + udy)}$$

(1) Let $\underline{F} = (u, -v)$

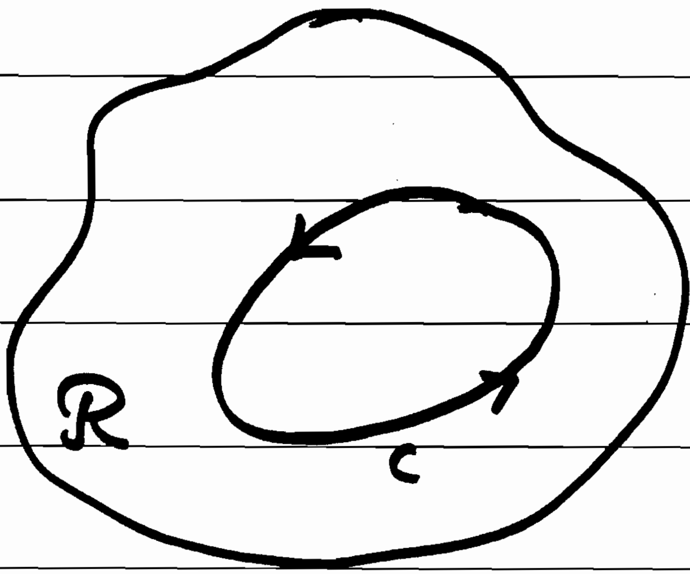
$$\nabla \times F = \hat{z}(-v_x - u_y) = 0 \quad (\text{since CR})$$

(2) Let $\underline{G} = (v, u) \Rightarrow \nabla \times G = \hat{z}(u_x - v_y) = 0$

Then

$$\oint (udx - vdy) = \iint \nabla \times F \cdot dS \equiv 0$$

$$\oint (vdx + udy) = \iint \nabla \times G \cdot dS \equiv 0$$



Simply connected $\mathcal{R}$ (theorem true even if we do not

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Cauchy Integral formula: $f(z)$ as before

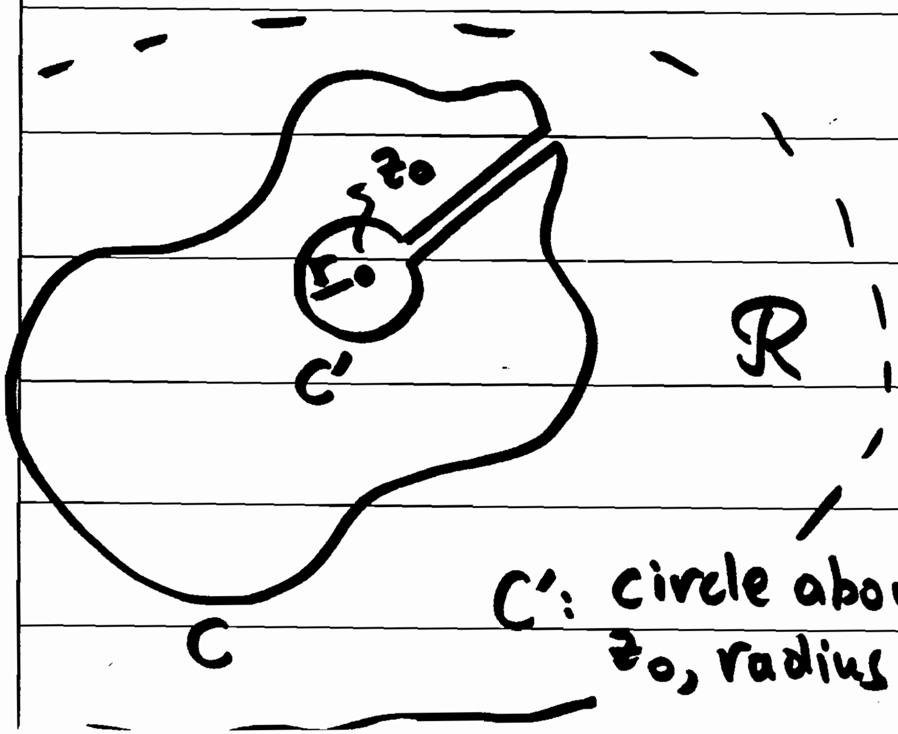
$$I = \oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C'} \frac{f(z)}{z-z_0}\,dz$$

on C': $z = z_0 + re^{i\theta}$; $dz = rie^{i\theta}d\theta$

$$I = \int \frac{f(z_0+re^{i\theta})}{\cancel{re^{i\theta}}}\, i\cancel{re^{i\theta}}\,d\theta = 2\pi i f(z_0)$$

(independent of $r \to$ let $r \to 0$)

$= 0$ if z_0 outside C



C': circle about z_0, radius r

Derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Derivatives: $n = 1$.

$$\frac{f(z_0+\delta z_0)-f(z_0)}{\delta z_0} = \frac{1}{2\pi i\,\delta z_0}\oint\left(\frac{f(z)}{z-z_0-\delta z_0} - \frac{f(z)}{z-z_0}\right)dz$$

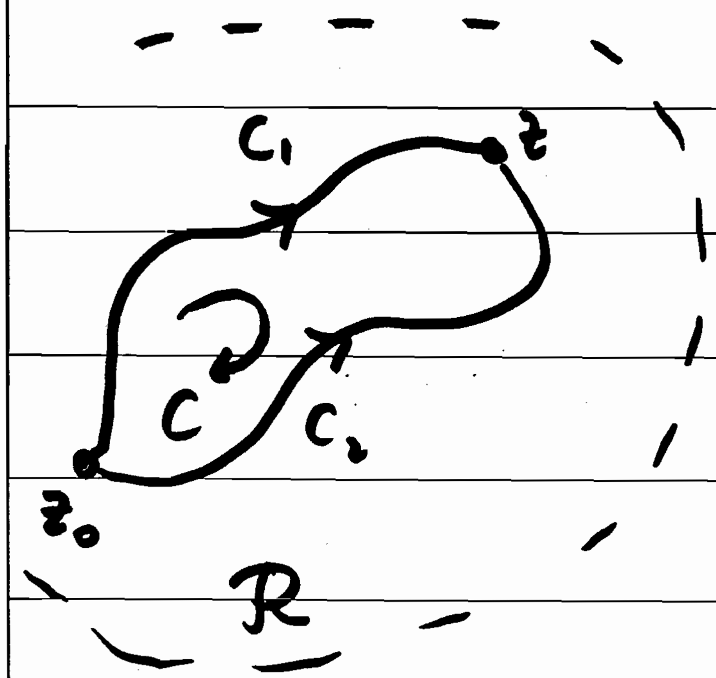
$$= \frac{1}{2\pi i\,\cancel{\delta z_0}}\oint \frac{\cancel{\delta z_0}\, f(z)\,dz}{(z-z_0)(z-z_0-\delta z_0)} \xrightarrow{\delta z_0 \to 0} \frac{1}{2\pi i}\oint \frac{f\,dz}{(z-z_0)^2}$$

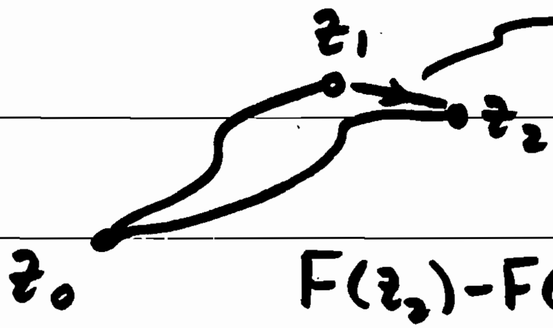
Antiderivatives: f analytic in $\mathcal{R}$

$$\oint_C f\,dz = 0 \Rightarrow \int_{C_1} f\,dz = \int_{C_2} f\,dz = \int_{z_0}^{z} f\,dz$$

integral independent of the path:

$$F(z) := \int_{z_0}^{z} f\,dz \quad . \quad \text{Fix } z_0$$





$z = z_1 + t(z_2 - z_1)$

$0 \le t \le 1$

$$F(z_2) - F(z_1) = \int_{2}^{z_1} f(z)\,dz$$

$$F(z_2)-F(z_1) = (z_2-z_1)\int_{t=0}^{1} f(t)\,dt$$

$$f(t) \equiv f(z_1+t(z_2-z_1)); \quad t \text{ real.}$$

Then $$\frac{F(z_2)-F(z_1)}{z_2-z_1} - f(z_1) =$$

$$= \underbrace{\int_{t=0}^{1} f(z_1+t(z_2-z_1))\,dt}_{f(z^*)\cdot 1} - f(z_1) \xrightarrow{z_2\to z_1} 0$$

$f(z^*)\cdot 1$, z^* between z_2, z_1
(mean value theorem)

i.e. $f(z_1) = F'(z_1)$, $F(z)$ differentiable in $\mathcal{R}$
$\Rightarrow F(z)$ analytic

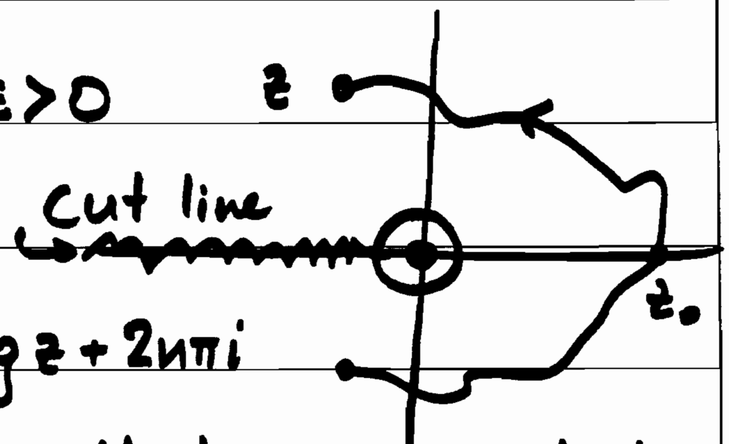
(*)[1] We did not use fact that f analytic – only that it is <u>continuous</u> and that $\int f\,dz$ is path independent.

⊛ Just <u>analyticity</u> does not suffice to give antiderivative:

Let $\mathcal{R}: |z| \geq \varepsilon > 0$

$f(z) = \frac{1}{z}$

$F(z) = \ln z = \text{Log}\, z + 2n\pi i$



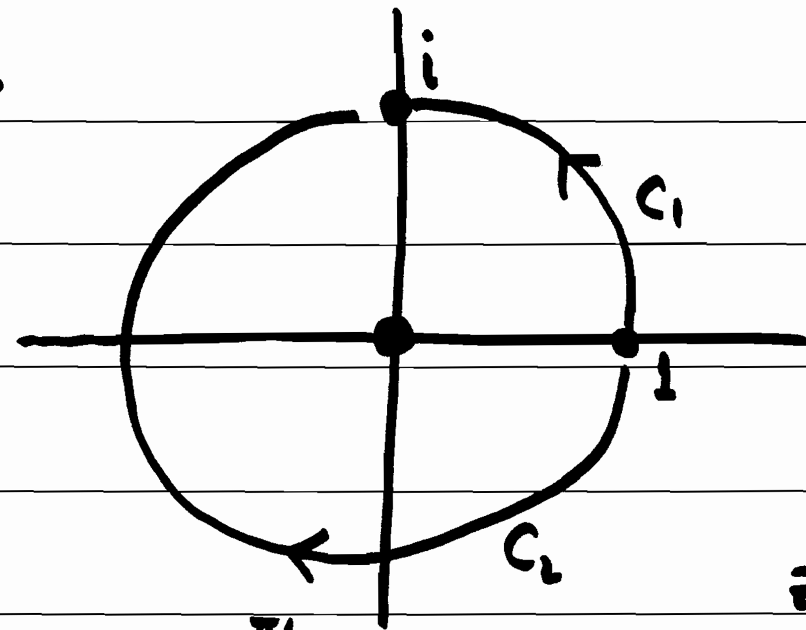
<u>Problem</u>: region multiply connected

If we cut at negative real axis, get simply connected..

<u>Morera's thm</u>: If $f(z)$ is continuous and its integral is independent of the path (i.e. $\oint_C f\, dz$ is $= 0$ for any closed path C in $\mathcal{R}$), then f is analytic in $\mathcal{R}$.

/// If f has antiderivative then it is analytic. For the converse, $\mathcal{R}$ must be simply connected! ///

*Ex



$z = re^{i\theta}$

$dz = ire^{i\theta}d\theta$

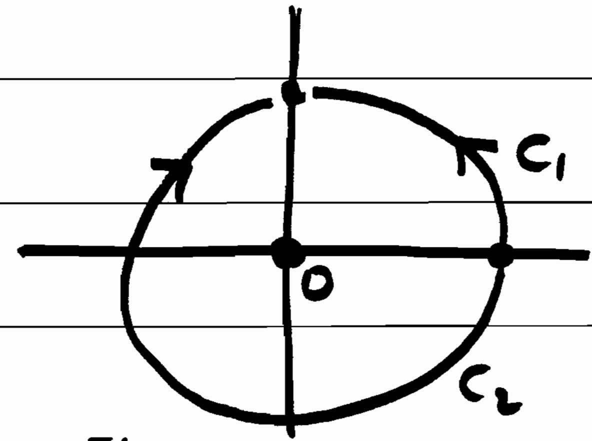
$$\int_{C_1} \frac{dz}{z} = \int_0^{\pi/2} \frac{i\cancel{re^{i\theta}}}{\cancel{re^{i\theta}}} d\theta = \frac{\pi}{2} i$$

$$\int_{C_2} \frac{dz}{z} = i\int_0^{-3\pi/2} d\theta = -\frac{3\pi}{2} i$$

Reason: $\ln i = \frac{\pi}{2} i + 2n\pi i$; $C_1 : n = 0$, $C_2 : n = -1$

$\ln z$ multivalued ; $f(z) = \frac{1}{z}$ not analytic at $z = 0$.

Ex: $f(z) = \dfrac{1}{z^2} \qquad F(z) = -\dfrac{1}{z}$



$$\int_{C_1} \frac{dz}{z^2} = \int_0^{\pi/2} \frac{ire^{i\theta}\,d\theta}{r^2 e^{2i\theta}} = \frac{i}{r}\int_0^{\pi/2} e^{-i\theta}\,d\theta$$

$$= -\frac{1}{r} e^{-i\theta}\Big|_0^{\pi/2} = -\frac{1}{r}\left(\frac{1}{i} - 1\right)$$

$$= \frac{1+i}{r}$$

$$\int_{C_2} \frac{dz}{z^2} = \cdots = -\frac{1}{r}\left(e^{-i\theta}\Big|_0^{-3\pi/2}\right) = -\frac{1}{r}\left(\frac{1}{i} - 1\right) = \frac{1+i}{r}$$

Independent of path, even though $\mathcal{R}$ not simply connected.

(sufficient vs necessary conditions)

Facts about analytic functions:

* $f(z)$ analytic around z_0: Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \oplus$$

$$a_n = \frac{1}{n!} \frac{d^n f(z_0)}{dz_0^n}$$

* If R is distance from z_0 to nearest singularity of $f(z)$, then $\oplus$ converges uniformly (i.e. rate is independent of z) in disk $D_r: |z-z_0| \leq r < R$.
* In $D_r(z_0)$, Taylor series can be differentiated and integrated term-by-term
* If $f(z)$ is entire, Taylor series converges in $\mathbb{C}$.

(Prove these later).

$(z_0 = 0)$

Conversely, there is R such that

$f(z) = \sum_0^\infty a_n z^n$ is analytic and

bounded for $|z| \le r < R$.

* If $|f(z)| \le M$ on $|z| = r$, then

$$|a_n| r^n \le M$$

◀ $M(r) = \max_{|z|=r} |f(z)|$

$$|a_n| = \frac{1}{2\pi} \left| \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \le \frac{M(r)}{2\pi} \cdot \frac{2\pi r}{r^{n+1}} = \frac{M(r)}{r^n} \quad ▶$$

$$\| $$

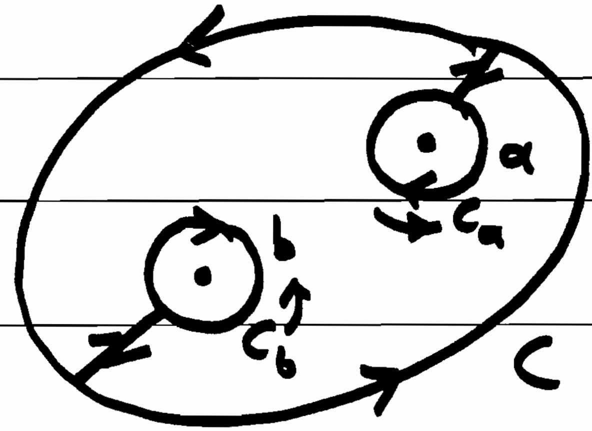
$$\left| \frac{1}{n!} f^{(n)}(0) \right|$$

<u>Liouville</u>: $f(z)$ entire, bounded $\Rightarrow f(z) =$ constant

◀ Above, let $r \to \infty$: $|a_n| \le \frac{M(r)}{r^n} \le \frac{M}{r^n} \xrightarrow{r \to \infty} 0$, $n \ge 1$

So $f(z) = a_0$ constant ▶

Problem: Evaluate $\int_C \frac{dz}{(z-a)(z-b)}$ with $\underline{a, b \text{ not on } C}$.



Partial fractions: $\frac{1}{(z-a)(z-b)} = \frac{1}{b-a}\frac{1}{z-a} - \frac{1}{b-a}\frac{1}{z-b}$

$$\oint_C = \oint_{C_a} + \oint_{C_b} \ ;$$

(1) The function $f_1(z) = \frac{1}{z-b}$ is analytic incide C_a.

So use Cauchy integral formula:

$$f_1(a) = \frac{1}{2\pi i} \oint_{C_a} \frac{f_1(z)}{z-a}\, dz$$

$$= \frac{1}{2\pi i} \oint_{C_a} \frac{dz}{(z-a)(z-b)}$$

$$\therefore \oint_{C_a} \frac{dz}{(z-a)(z-b)} = 2\pi i \frac{1}{a-b}$$

Similarly: $f_2(z) = \frac{1}{z-a}$ is analytic inside C_b:

$$f_2(b) = \frac{1}{2\pi i} \oint_{C_b} \frac{f_2\, dz}{z-b} = \frac{1}{2\pi i} \oint_{C_b} \frac{dz}{(z-a)(z-b)} \Rightarrow$$

$$\oint_{C_b} \frac{dz}{(z-a)(z-b)} = 2\pi i \cdot \frac{1}{b-a} \Rightarrow \oint_{C} \frac{dz}{(z-a)(z-b)} = 0$$

if both a, b inside C