Isolated Singularities

z_0: Laurent series in $0<|z-z_0|<R$

$$f(z) = \sum_{k=n}^{\infty} a_k (z-z_0)^k \quad ; \; a_n \neq 0$$

$n>0$: z_0 is zero of order n ($\underset{f}{\text{analytic}}$)

$n<0$: " pole " " n ($|f| \xrightarrow{z\to z_0} \infty$)

$n=-\infty$: essential singularity

(f : zero of order n at z_0 $\rightarrow$ $1/f$: pole of order n at z_0)

<u>Ex</u>: $f(z) = e^z - 1 = z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n$ = zero of order 1

$$= z\left(1 + \frac{1}{2!}z + \cdots + \frac{1}{n!}z^{n-1} + \cdots\right)$$

$$\frac{1}{f(z)} = \frac{1}{e^z-1} = \frac{1}{z}\,\frac{1}{1+\frac{1}{2}z+\cdots} = \frac{1}{z}\left(b_0 + b_1 z + \cdots\right)$$; pole of order 1

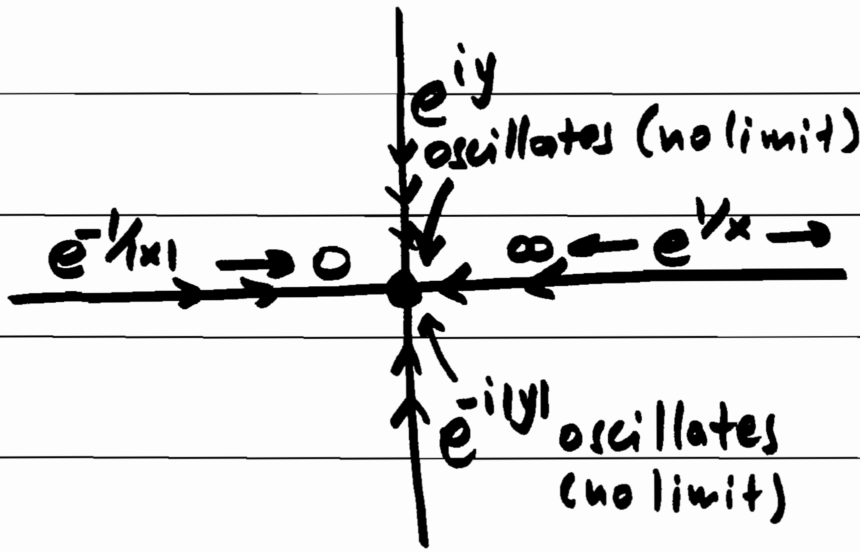
Essential singularity:

(isolated; Laurent series has ∞ neg. order terms).

__Note__: this only applies to Laurent series valid in $0<|z-z_0|$

Ex: $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \dots + \frac{1}{n!}\frac{1}{z^n}$,

as $z \to 0$:



__Pickard thm__: in the neighborhood of an essential singularity, a function comes arbitrarily close to any value.

Ex: $f(z) = \tan\left(\frac{1}{z}\right) = \dfrac{e^{i/z} - e^{-i/z}}{e^{i/z} + e^{-i/z}}$

$z = 0$: essential singularity? No!

Note that $\cos\left(\frac{1}{z}\right) = 0 \Rightarrow \frac{1}{z} = (k+\frac{1}{2})\pi$

$\Rightarrow z = \dfrac{1}{(k+\frac{1}{2})\pi}, \quad k = 0, \pm 1, \pm 2, \cdots$

These points are order 1 zeroes for $\cos z$

(since ~~cos z = 1~~ $\cos(z - z_0) = \cos z \cos z_0 - \sin z \sin z_0$

For $z_0 = (k+\frac{1}{2})\pi$, $\cos z_0 = 0$, $\sin z_0 = (-1)^k$, so

$\cos(z - z_0) = (-1)^{k+1} \sin z = (-1)^{k+1}\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right)$; zero, order 1

$\Rightarrow f(z)$ has poles of order 1 at $z_{0k} = \left[(k+\frac{1}{2})\pi\right]^{-1}$. But

as $k \to \infty$ $z_{0k} \to 0$: Then $z = 0$ not isolated!

Behavior at $z = \infty$:

Replace z by $\zeta = \frac{1}{z}$, examine behavior as $\zeta \to 0$

Ex: examine $f(z) = z + z^3$ as $z \to \infty$

Let $g(\zeta) = f\left(\frac{1}{\zeta}\right) = \frac{1}{\zeta} + \frac{1}{\zeta^3}$. This function has a pole of order 3 at $\zeta = 0$.

Ex: $\tan\left(\frac{1}{z}\right) = \tan \zeta$; has zero of order 1 at $\zeta = 0$.

Ex: $f(z) = e^z \Rightarrow g(\zeta) = f\left(\frac{1}{\zeta}\right) = e^{1/\zeta}$ has essential singularity at $\zeta = 0$.

Ex: $\dfrac{1}{1+z} = \dfrac{1}{1 + 1/\zeta} = \dfrac{\zeta}{\zeta + 1} \xrightarrow{\zeta \to 0} 0$

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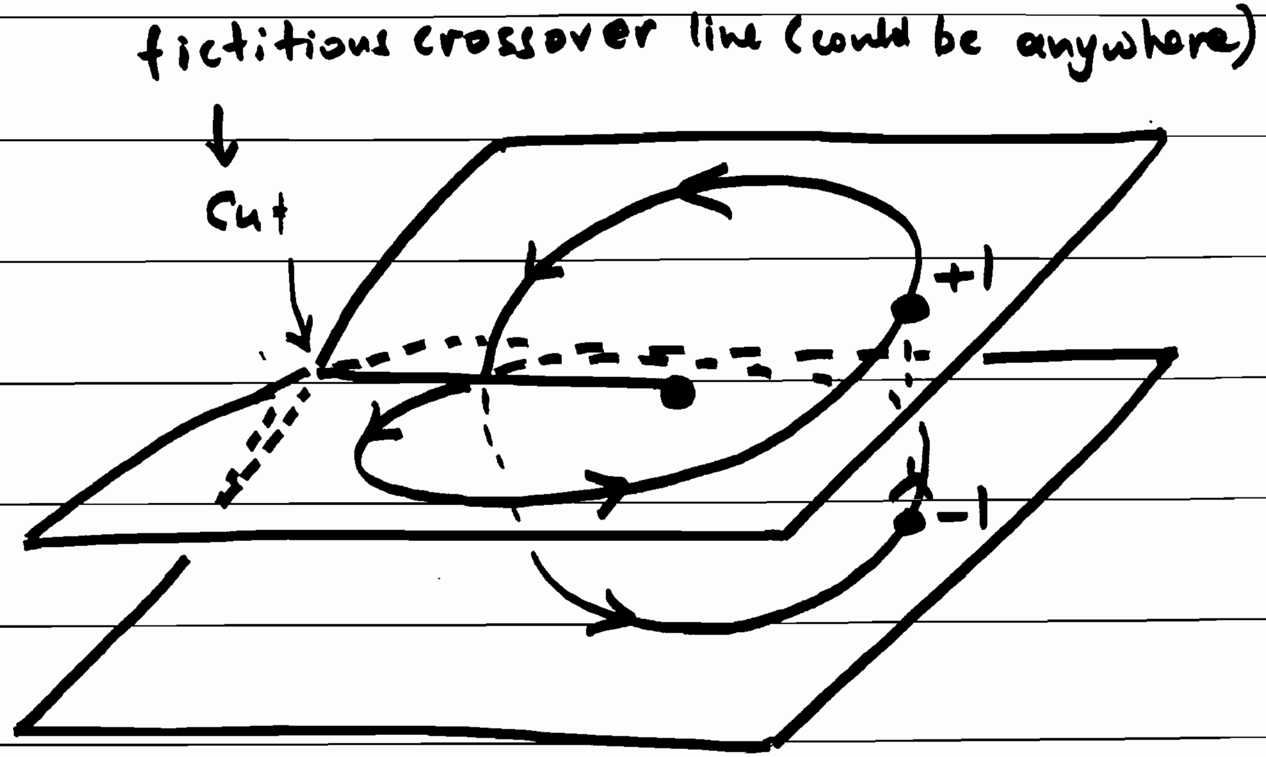
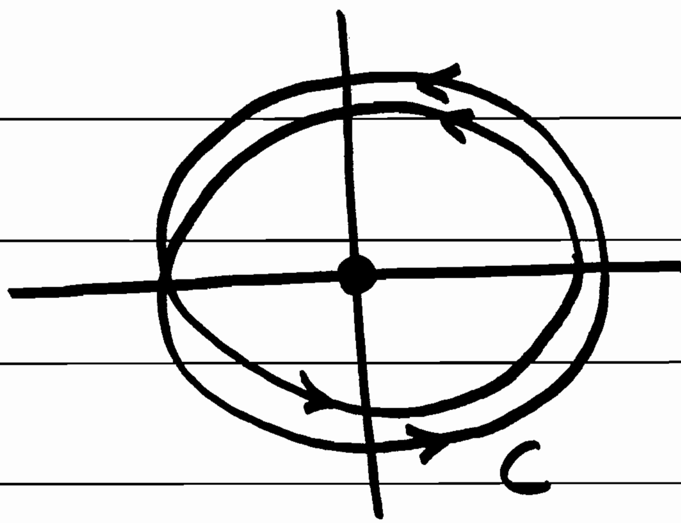




$C: r=1, \; 0 \le \theta \le \theta_0$

$$\int_C \frac{dz}{\sqrt{z}} = 2\sqrt{z}\,\Big|_0^{\theta_0} = 2e^{i\theta/2}\Big|_0^{\theta_0}$$

$$= 2\left(e^{i\theta/2} - 1\right) \neq 0$$

for $\theta = 2\pi$, although full circle and function has "antiderivative": multivalued.

(Will come back after two circuits)

Ex: series for root-functions

$f(z) = \sqrt{z}$; find Taylor series about $z = 1$:

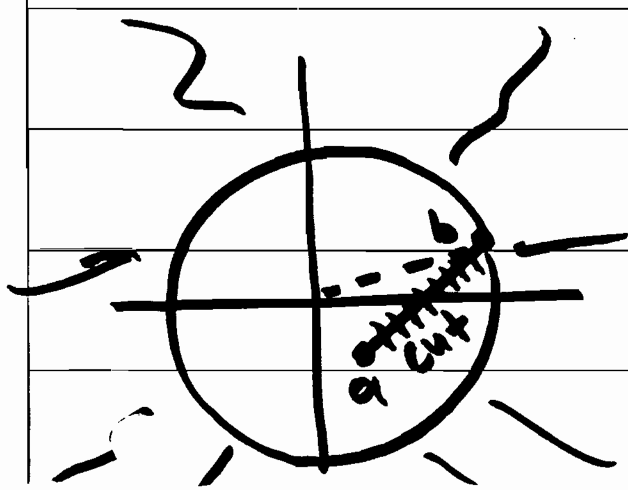
$$\sqrt{z} = (1 + (z-1))^{1/2} = (1+\zeta)^{1/2} \; ; \quad |\zeta| < 1$$

$$= \pm\left(1 + \tfrac{1}{2}\zeta - \tfrac{1}{8}\zeta^2 + \dots\right) \quad \text{(two branches)}$$

Ex: $f(z) = \sqrt{(z-a)(z-b)}$; find Laurent series around $z=0$, valid for large z: introduce

$$\zeta = \frac{1}{z} \; ; \quad g(\zeta) = f\left(\frac{1}{\zeta}\right) = \frac{1}{\zeta}(1-a\zeta)^{1/2}(1-b\zeta)^{1/2}$$

find series valid for small ζ: $0 < |\zeta| < \frac{1}{|b|}$



$$g(\zeta) = \pm\frac{1}{\zeta}\left(1 - \frac{a}{2}\zeta + \dots\right)\left(1 - \frac{b}{2}\zeta + \dots\right)$$

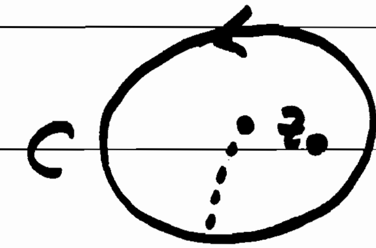
$$= \pm\frac{1}{\zeta}\left(1 - \frac{a+b}{2}\zeta + \dots\right) \quad \text{(two possibilities)}$$

$z = \infty$: pole of order 1

Poles and Residues : z_0 isolated sing.

$$f(z) = \sum_{m}^{\infty} a_k (z-z_0)^k \quad , 0<|z-z_0|<R \ (\le \infty)$$

$$\underline{\overline{\mathrm{Res}\, f(z_0) = a_{-1}}}$$



only singularity of $f(z)$ in C

Significance:

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

Set $z_0 = 0$

(1) Simple pole: $f = \frac{a_{-1}}{z} + a_0 + \cdots \Rightarrow a_{-1} = \lim_{z\to 0} (z f(z))$

(2) Pole of order m: $f = \frac{a_{-m}}{z^m} + \cdots + \frac{a_{-1}}{z^{-1}} + a_0 + \cdots$

$\underset{\text{analytic}}{z^m f} = a_{-m} + \cdots + a_{-1} z^{m-1} + a_0 z^m + \cdots$

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[z^m f(z) \right]_0 = a_{-1}$$

* no such method for essential singularities (must find a_{-1} directly)

Ex. $f(z) = \dfrac{1-e^{2z}}{z^4}$ pole : $z=0$

$$\frac{1}{z^4}\left(1-e^{2z}\right) = \frac{-1}{z^4}\left(2z+2z^2+3z^3+\frac{2}{15}z^4+\dots\right)$$

$$= -\frac{2}{z^3} \mp \frac{2}{z^2} - \frac{3}{z} + \frac{2}{15} + \cdots$$

Res $(z=0) = -3$; pole of order 3.

Ex: $f(z) = \sec z = \dfrac{1}{\cos z}$; simple poles

at $z_k = (k+\frac{1}{2})\pi$

$$\text{Res}\, f(z_k) = \lim_{z\to z_k} \frac{z-z_k}{\cos z} = -\frac{1}{\sin z_k} = -(-1)^k \quad \text{(l'Hospital rule)}$$

Ex: $f(z) = \dfrac{3z+2}{z^4+1}$; simple poles where $z^4=-1$, $z_k = e^{\frac{i(2k+1)\pi}{4}}$, $k=0,1,2,3$

$$\text{Res}\, f(z_k) = \lim_{z\to z_k} \frac{3z+2}{z^4+1}\cdot(z-z_k) = (3z_k+2)\cdot\underbrace{\frac{1}{4z_k^3}}_{\text{l'Hospital}} = \frac{3e^{i(2k+1)\pi/4}+2}{4e^{3i(2k+1)\pi/4}}$$

This is a very common calculation: must do efficiently

$$\lim_{z \to z_k} \left[\frac{z - z_k}{z^4 + 1} \cdot (3z+2) \right] = (3z_k + 2) \cdot \left. \frac{(z - z_k)'}{(z^4+1)'} \right|_{z_k}$$

first factor has limit; second gives $\frac{0}{0}$, so use l'Hospital's rule.

<u>Ex.</u> $f(z) = e^{1/z} \sin(1/z) \quad z = 0$

$$= e^{1/z}\left(e^{i/z} - e^{-i/z}\right)\frac{1}{2i} = \left[e^{(1+i)/z} - e^{(1-i)/z}\right]\frac{1}{2i}$$

$$= \frac{1}{2i}\left\{ 2 + \frac{(1+i)-(1-i)}{z} + \frac{(1+i)^2-(1-i)^2}{2z^2} + \dots \right\}$$

$$\operatorname{Res} f(0) = \frac{1+i-(1-i)}{2i} = 1$$