METHODS OF C-INTEGRATION

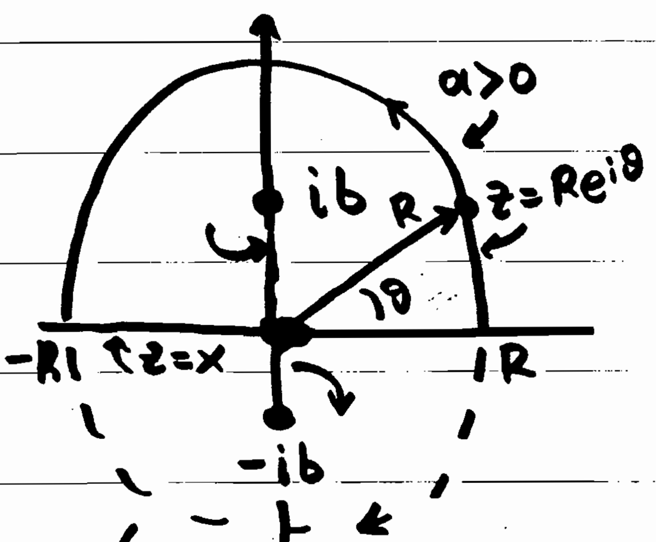
(1) $\int_{-\infty}^{\infty} f(x)\,dx$; $|f(z)| < \dfrac{M}{|z|^{1+\delta}}$ for $|z|>R_0$

continuation of $f(x)$ on either upper or lower halfplane

Ex. $I(a) = \displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}\,dx$; $a, b > 0$

$$I = \mathrm{Re}\, I_1 = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\,dx = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\,dx$$

(even)



$$\underbrace{\int_{-R}^{R} \frac{e^{iax}}{x^2+b^2}\,dx}_{I(a) \text{ as } R\to\infty} + \underbrace{\int_0^{\pi} \frac{e^{iaz}\, Rie^{i\theta}\,d\theta}{R^2 e^{2i\theta}+b^2}}_{\to 0}$$

$$= 2\pi i \cdot \mathrm{Res}(ib) = 2\pi i \cdot \frac{\overbrace{e^{iaz}}^{=e^{-ab}}}{\underbrace{z+ib}_{=2ib}}\Bigg|_{ib}$$

$$= \frac{\pi}{b} e^{-ab}$$

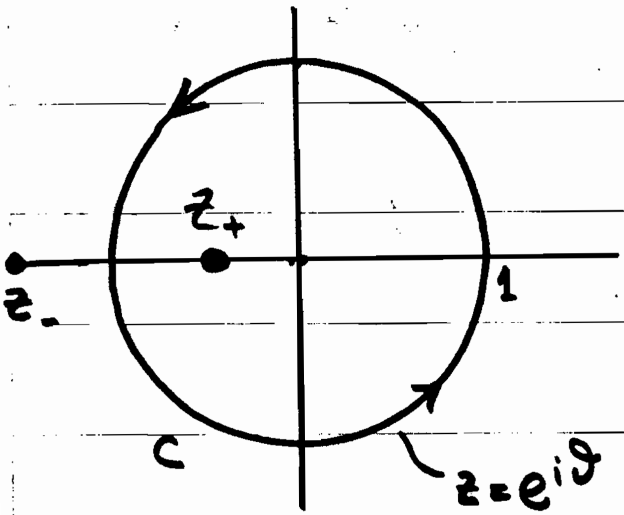
$$(2)\ I = \int_0^{2\pi} R(\cos\vartheta, \sin\vartheta)\, d\vartheta = \oint_{C}^{} R(\cos\vartheta, \sin\vartheta)\, \frac{dz}{iz}$$

$z = e^{i\vartheta}$; $dz = ie^{i\vartheta} d\vartheta = iz\, d\vartheta$, $\quad \frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$I = 2\pi i \sum \mathrm{Res}(\tilde{R}(z))$ where

$$\tilde{R}(z) = R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\cdot\frac{1}{z}$$

Ex: $I = \int_0^{2\pi} \frac{d\vartheta}{1+\alpha\cos\vartheta}$; $|\alpha| < 1$



$$I = \frac{1}{i}\oint_{|z|=1} \frac{dz}{z}\, \frac{1}{1+\frac{\alpha}{2}\left(z+\frac{1}{z}\right)} = \frac{2}{i}\oint_{|z|=1} \frac{dz}{\alpha z^2 + 2z + \alpha}$$

$z_\pm = -\frac{1}{\alpha} \pm \sqrt{\frac{1}{\alpha^2} - 1}$; $z_+ z_- = 1$, $\alpha \neq 1$

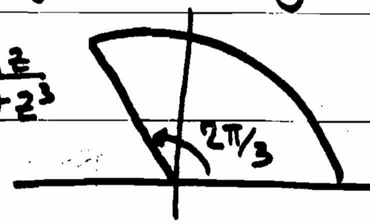
$z_+ = -\frac{1}{\alpha} + \sqrt{\frac{1}{\alpha^2} - 1}$ $\quad$ only one root inside (z_+)

$$I = \frac{2}{i}\cdot 2\pi i \cdot \mathrm{Res}(z_+) =$$

$$= 4\pi \frac{1}{\alpha(z_+ - z_-)} = \frac{2\pi}{\sqrt{1-\alpha^2}}$$

(3) $\int_0^\infty f(x)\,dx$ where $f(z)$ has symmetry

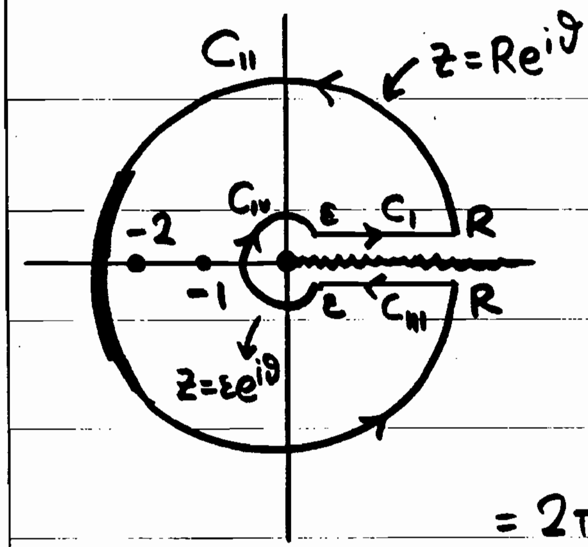
Ex: $\int_0^\infty \frac{dx}{1+x^3} \rightarrow \oint_c \frac{dz}{1+z^3}$



since $(re^{2\pi i/3})^3 = r^3$

This is not always possible!

Ex. $I = \int_0^\infty \frac{dx}{x^2+3x+2}$



Consider

$B = \oint_c \frac{\ln z\,dz}{z^2+3z+2}$ Show $C_{II}, C_{IV} \rightarrow 0$

Then $\int_{C_I} - \int_{C_{III}} = 2\pi i \sum \text{Res} = -2\pi i\, I$ (see next p.)

$= 2\pi i \left(\frac{\ln(2e^{i\pi})}{-2+1} + \frac{\ln(e^{i\pi})}{-1+2} \right) = -2\pi i \ln 2$

$\qquad\quad z+1\big|_{z=-2} \qquad z+2\big|_{z=-1} \qquad \Rightarrow I = \ln 2$

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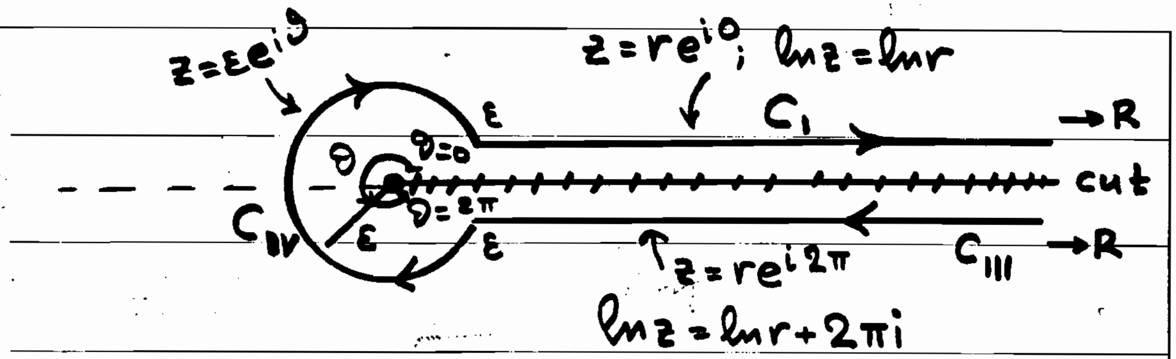


$$\text{on } C_I: \quad \frac{\ln z}{z^2+3z+2} = \frac{\ln r + 0i}{r^2+3r+2}$$

$$\text{on } C_{III}: \quad \frac{\ln z}{z^2+3z+2} = \frac{\ln r + 2\pi i}{r^2+3r+2}$$

$$\underset{\xrightarrow[(+)]{}}{\int_{C_I}} + \underset{\xleftarrow[(-)]{}}{\int_{C_{III}}} = \int_\varepsilon^R \frac{\ln r - (\ln r + 2\pi i)}{r^2+3r+2}\,dr = -2\pi i \int_\varepsilon^R \frac{dr}{r^2+3r+2} \xrightarrow[\varepsilon\to 0]{R\to\infty} -2\pi i\, I$$

$$C_{IV}: z=\varepsilon e^{i\vartheta}: \left|\int_{C_{IV}} \cdots\right| \le \int_{C_{IV}} \frac{|\ln\varepsilon + i\vartheta|\,|\varepsilon i e^{i\vartheta}|\,d\vartheta}{|\varepsilon^2 e^{2i\vartheta} + 2\varepsilon e^{i\vartheta} + 3|} \le \int_0^{2\pi} \frac{(|\ln\varepsilon| + |\vartheta|)\varepsilon\, d\vartheta}{3 - 2\varepsilon - \varepsilon^2} \xrightarrow{\varepsilon\to 0} 0$$

$$C_{II}: z = Re^{i\vartheta}: \left|\int_{C_{II}} \cdots\right| \le \int \frac{(|\ln R| + |\vartheta|)\, R\, d\vartheta}{R^2 - 3R - 2} \xrightarrow{R\to\infty} 0$$

$$\varepsilon \ln\varepsilon \xrightarrow{\varepsilon\to 0} 0, \quad \frac{\ln R}{R} \xrightarrow{R\to\infty} 0$$

In general, if $|f(z)| < \frac{M}{R^{1+\delta}}$, $\underset{R}{\overset{\shortparallel}{|z|}} > R_0$

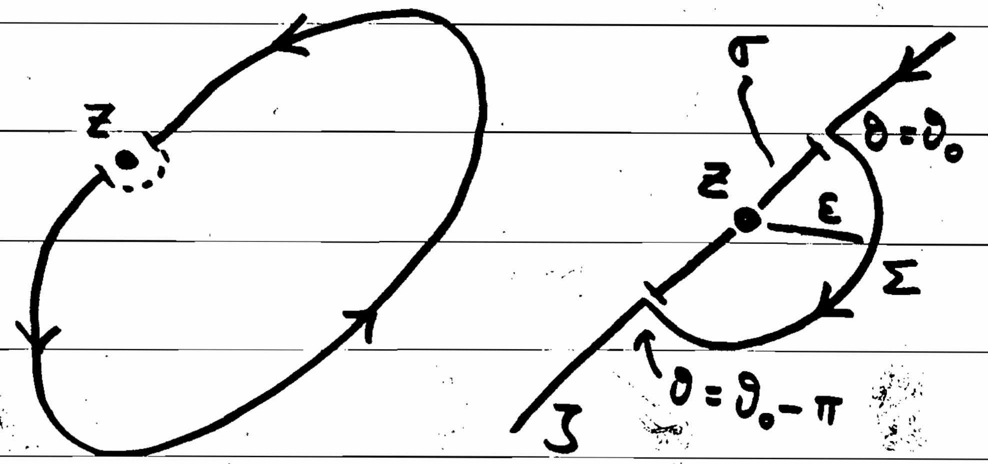
then can evaluate

$\int_0^\infty f(x)\,dx$ by key-hole contour for

$\oint_c^\infty \ln z\, f(z)\,dz$.

<u>Note</u>: above we assumed that $f(z)$ (the continuation of $f(x)$ for complex values) had no poles on the (+) real axis. In fact we can also allow simple poles on the real axis. For that we need to introduce the

"Cauchy Principal Value"



$$I = \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \lim_{\varepsilon \to 0} \int_{C-\sigma} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Cauchy principal value, let "lips" of excluded interval close at the same rate.

let $C' = C - \sigma$; then

$$\oint_{C' + \Sigma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0 \quad \text{(no poles inside).}$$

And: $\lim_{\epsilon \to 0} \oint_{C'} \equiv$ P.V. $\oint_C$ (desired integral)

But $$\lim_{\varepsilon \to 0} \int_\Sigma \frac{f(\zeta)}{z - \zeta}\, d\zeta = f(z) \lim_{\varepsilon \to 0} \int_{\theta_0}^{\theta_0 - \pi} \frac{i \cancel{\varepsilon e^{i\theta}}\, d\theta}{\cancel{\varepsilon e^{i\theta}}} = -\pi i f(z)$$

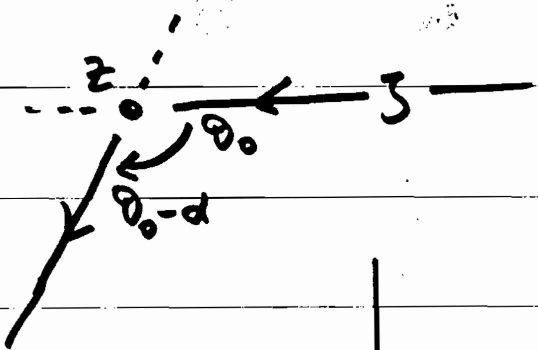
$\zeta = z + \varepsilon e^{i\theta}$, $\theta_0 - \pi \le \theta \le \theta_0$ $\quad f(\zeta) = f(z + \varepsilon e^{i\theta}) \xrightarrow[\varepsilon \to 0]{} f(z)$

So: $\displaystyle PV \oint \frac{f(\zeta)}{\zeta - z}\, d\zeta = \pi i f(z)$

(instead of $2\pi i f(z)$ if z was interior)

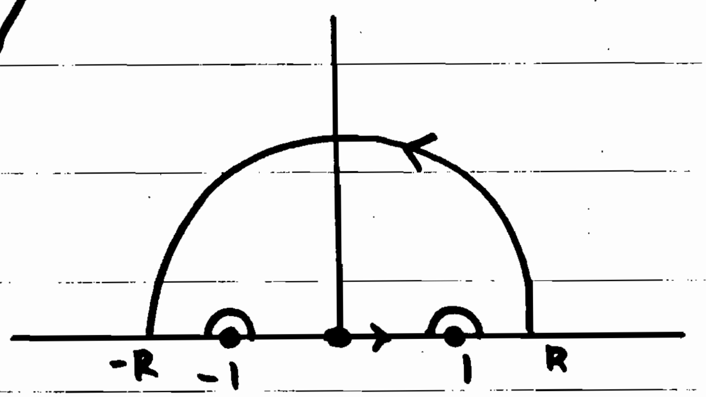
<u>Note:</u> if contour had an angle α at z (i.e. it was not smooth), then

$$PV \oint \frac{f(\zeta)}{\zeta - z}\, d\zeta = i\alpha f(z)$$



<u>Ex</u>: $\displaystyle PV \int_{-\infty}^{\infty} \frac{dx}{x^2-1} = \pi i \left(\frac{1}{2} \frac{1}{z+1}\Big|_{z=1} + \frac{1}{2} \frac{1}{z-1}\Big|_{z=-1} \right)$

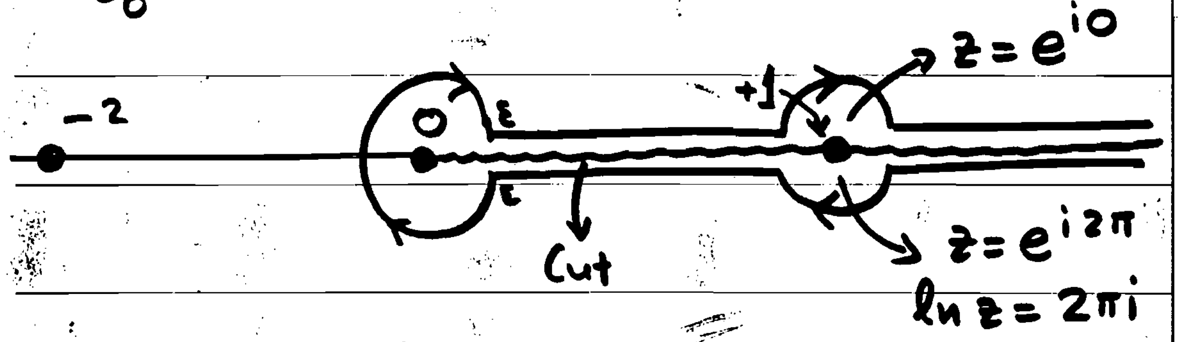
$= 0$



Ex: (simple pole on the branch cut)

$$I=\int_0^\infty \frac{dz}{z^2+z-2} \quad : \quad \oint \frac{\ln z \, dz}{z^2+z-2}$$

$\ln z = 0$



Work as before. But now

$$-2\pi i\, I = 2\pi i\left(\mathrm{Res}(z=-2) + \frac{1}{2}\mathrm{Res}(z=e^{i0}) + \frac{1}{2}\mathrm{Res}(e^{i2\pi})\right)$$

note (+) sign

$$\mathrm{Res}(e^{i0}) = \left.\frac{\ln z}{z+2}\right|_{z=e^{i0}} = \frac{0}{3}$$

$$\mathrm{Res}(e^{i2\pi}) = \left.\frac{\ln z}{z+2}\right|_{z=e^{i2\pi}} = \frac{2\pi i}{3}$$

$$\mathrm{Res}(-2=2e^{i\pi}) = \left.\frac{\ln z}{z-1}\right|_{z=2e^{i\pi}} = \frac{\ln 2 + i\pi}{-3}$$

$$I = -\left(\frac{\ln 2 + i\pi}{-3} + 0 + \frac{1}{2}\left(\frac{2\pi i}{3}\right)\right) = \ln 2/3$$

$$(5)\ I = \int_{-\infty}^{\infty} e^{iax} f(x)\,dx \quad , a > 0$$

with $|f(z)| < \dfrac{M}{|z|}$, for $|z| > R_0$

$$I = \lim_{R\to\infty}\left(\int_{-R}^{R} + \int_{C_R}\right) = 2\pi i \sum \left(\begin{array}{c}\text{Res}\\ \text{in upper}\\ \text{half-plane}\end{array}\right)$$

(if $a<0$, complete in lower half).

<u>Jordan lemma:</u> $\left|\int_{C_R} f e^{iaz} dz\right| \xrightarrow{R\to\infty} 0$

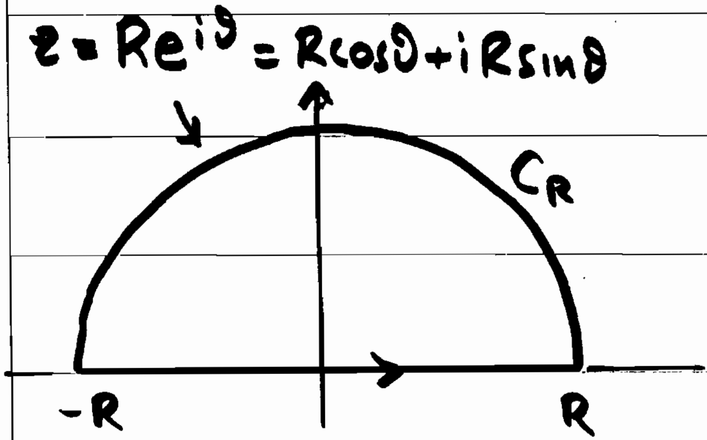
$$\left|\int_{C_R} \cdots\right| \leq \int_{C_R} |f(Re^{i\theta})| \cdot e^{-aR\sin\theta} R\,d\theta < M \int_0^{\pi} e^{-aR\sin\theta} d\theta$$

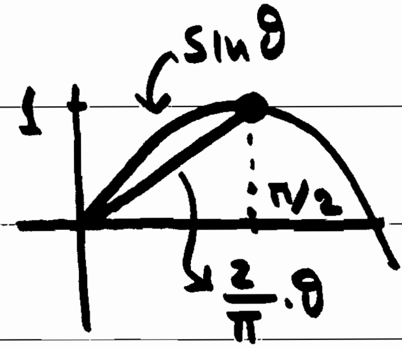
$$= 2M \int_0^{\pi/2} e^{-aR\sin\theta} d\theta$$

$$\leq 2M \int_0^{\pi/2} e^{-aR\cdot\frac{2\theta}{\pi}} d\theta =$$

$$\frac{M\pi}{aR}\left(1 - e^{-aR}\right) \xrightarrow{R\to\infty} 0$$

if $a > 0$.





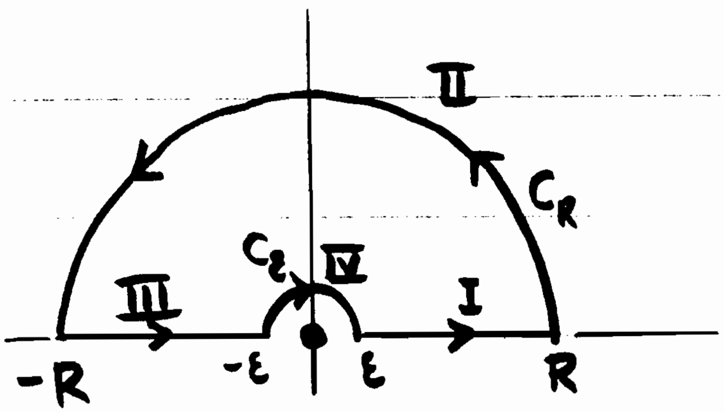
$$\sin\theta \geq \frac{2}{\pi}\theta \ , \ 0 \leq \theta \leq \frac{\pi}{2}$$

Ex $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,dx$, $a>0$ (Principal value)

Consider $\displaystyle\oint \frac{e^{iaz}}{z}\,dz = PV\int_{-R}^{R} ''' + \int_{C_R} ''' + \int_{C_\varepsilon} = 0$ (no poles inside)

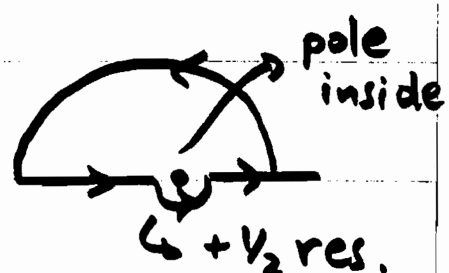
Since $\displaystyle\lim_{R\to\infty}\int_{C_R} = 0$ by Jordan lemma,

$$PV\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,dx = \pi i \,\mathrm{Res}(z=0) = \pi i$$



i.e. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$

(what if we used:



Not : C_ε contributes $-\frac{1}{2}$ the re due at $z=0$

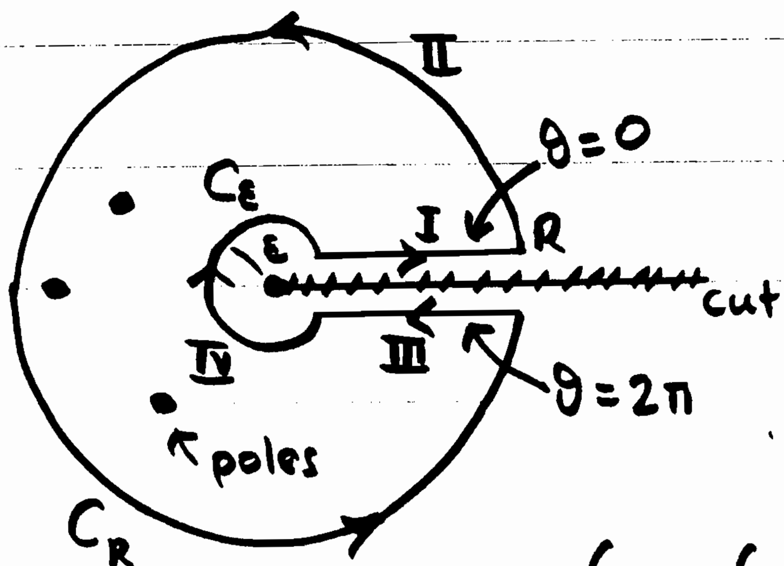
(6) $I = \int_0^\infty x^{a-1} f(x)\,dx$, $0<a<1$

$|f(z)| \to 0$ as $|z| \to \infty$; only poles ("meromorphic").

Consider $\oint z^{a-1} f(z)\,dz = 2\pi i \sum \text{Res}(\ldots)$ all poles

(if poles on + real axis, take principal values, get ½ residue from each side of branch cut). Can show:

(i) $C_I : \int_\varepsilon^R \cdots \to I$ as $R\to\infty$, $\varepsilon\to 0$



(ii) on C_{III}, $z = re^{i2\pi}$, $z^{a-1} = r^{a-1} e^{i2\pi(a-1)}$

$\int_{III} \to -e^{i2\pi(a-1)}\, I$

while $\int_{\ldots}, \int_{\ldots} \to 0$ as $R\to\infty$, $\varepsilon\to 0$

so: $\oint_{\text{III}} \longrightarrow \left(1-e^{i2\pi(a-1)}\right) I = 2\pi i \sum(\text{Res}).$

Ex. $\displaystyle\int_0^\infty \frac{x^{a-1}}{x+1}dx$: $\displaystyle\oint \frac{z^{a-1}}{z+1}dz =$

$= 2\pi i \cdot z^{a-1}\Big|_{z=e^{i\pi}} = 2\pi i\, e^{i\pi(a-1)}$

pole: $z=-1$

(I) $\displaystyle\int_\varepsilon^R \longrightarrow I$

(III) $\displaystyle\int_R^\varepsilon \frac{(re^{i2\pi})^{a-1}\overbrace{e^{i2\pi}}^{1}dr}{\underbrace{re^{i2\pi}}_{1}+1} \Rightarrow -e^{i2\pi(a-1)} I$

on II: $\displaystyle\left|\int_0^{2\pi} \cdots\right| \le \int_0^{2\pi} \frac{|Re^{i\theta}|^{a-1} R\,d\theta}{|Re^{i\theta}+1|} \le \int_0^{2\pi} \frac{R^{a-1}R}{R-1}d\theta = \frac{2\pi R^a}{R-1} \xrightarrow{R\to\infty} 0 \quad (a<1)$

on IV: $\displaystyle\left|\int_{2\pi}^0 \cdots\right| \le \int_0^{2\pi} \frac{|\varepsilon e^{i\theta}|^{a-1}\varepsilon\,d\theta}{|1+\varepsilon e^{i\theta}|} \le \frac{\varepsilon^a}{1-\varepsilon}\cdot 2\pi \xrightarrow{(\varepsilon\to 0)} 0$ $(a>0)$

So: $\left(1-e^{i2\pi(a-1)}\right) I = 2\pi i \cdot e^{i\pi(a-1)} \Rightarrow I = \dfrac{2\pi i\, e^{i\pi(a-1)}}{1-e^{i2\pi(a-1)}}$

$\Rightarrow I = \dfrac{\pi}{\sin \pi(1-a)}$

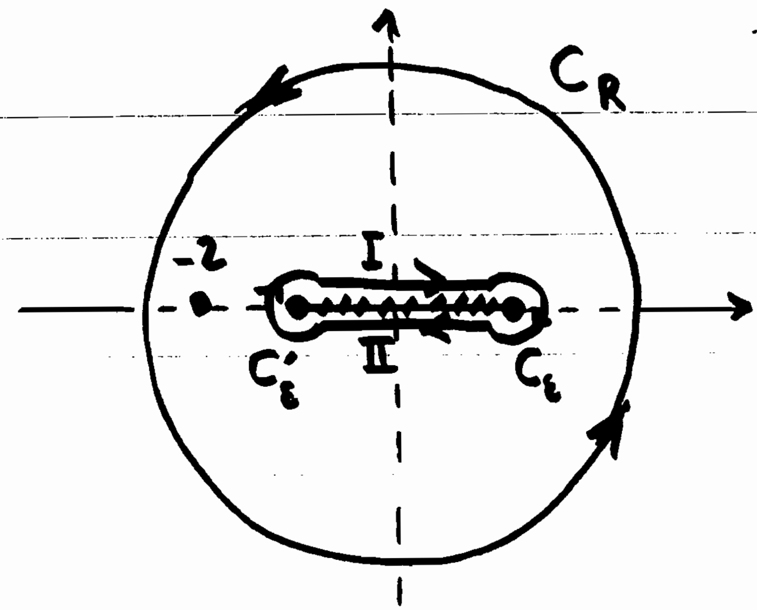
(7) Special problems

(a) $\displaystyle\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{1-x^2}}$ integrate around cut

$(|f| \to 0, \ |z| \to \infty)$

Ex. $\displaystyle I = \int_{-1}^{1} \frac{dx}{(x+2)\sqrt{1-x^2}}$

Consider $\displaystyle\oint \frac{dz}{(z+2)\sqrt{1-z^2}} =$



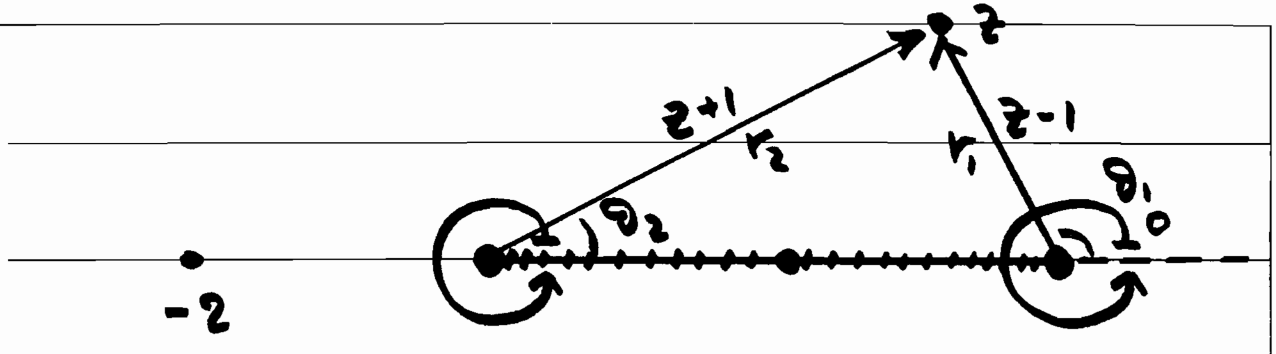
$$= 2\pi i \cdot \mathrm{Res}(-2) = 2\pi i \cdot \left.\frac{1}{\sqrt{1-z^2}}\right|_{-2}$$

$$= 2\pi i \cdot \frac{1}{\pm i\sqrt{3}} = \pm \frac{2\pi}{\sqrt{3}}$$

Problem: we must define sq. root so we get (+) value for I !

As $R \to \infty$, $\varepsilon \to 0$, integrals around $C_\varepsilon, C_\varepsilon', C_R \to 0$.

10.6/8



Need to define $f(z) = (1-z^2)^{-1/2}$ so that

$$f(-2) = +\frac{1}{i\sqrt{3}}$$

Now $(1-z^2)^{1/2} = \pm i(z^2-1)^{1/2} = \pm i(r_1 r_2)^{1/2} e^{i\left(\frac{\vartheta_1+\vartheta_2}{2}\right)}$

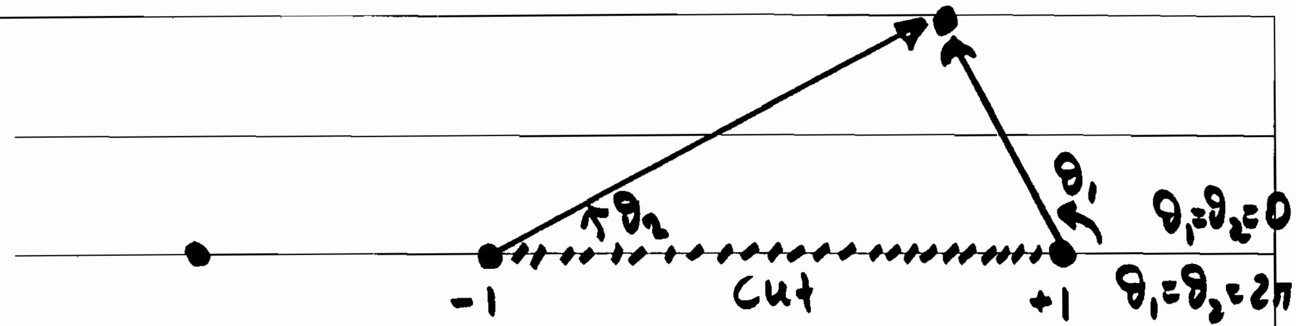
Choosing $0 \leq \vartheta_1, \vartheta_2 < 2\pi$ gives:

$$(1-z^2)^{1/2}\Big|_{-2} = \pm i\sqrt{3}\, e^{i\,2\pi/2} = \mp i\sqrt{3}$$

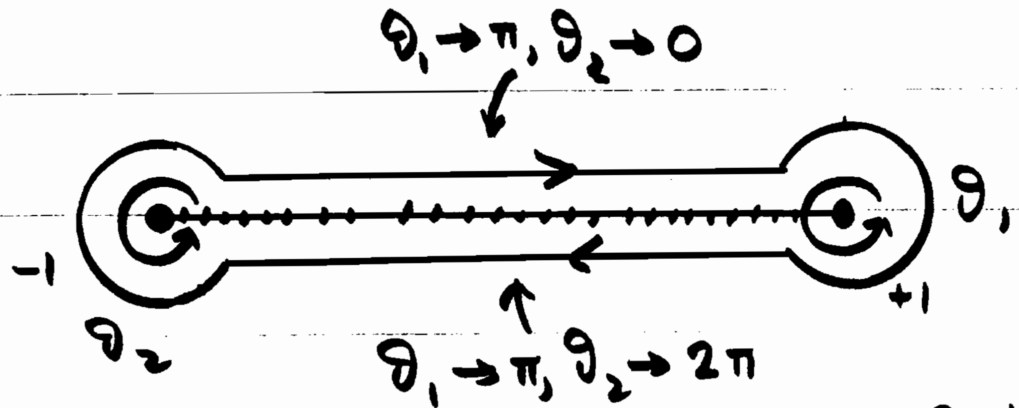
So, we will employ the definition

$$(1-z^2)^{1/2} = -i\,(z^2-1)^{1/2}, \quad 0 \leq \vartheta_1, \vartheta_2 < 2\pi$$

10.7/8



The cut extends only between $(-1, +1)$: since both θ_1, θ_2 jump by 2π, $f(z)$ is continuous on real axis, $x > +1$!



Top	Bottom
Top: $f(z) = -i\left(\lvert z-1\rvert \lvert z+1\rvert\right)^{1/2} e^{i\frac{\pi+0}{2}}$	$f(z) = -i\left(\lvert z-1\rvert \lvert z+1\rvert\right)^{1/2} e^{i\frac{\pi+2\pi}{2}}$
$= \sqrt{1-x^2}$ (+, real)	$= (-i)^2 \sqrt{1-x^2} = -\sqrt{1-x^2}$

10.8/8

So: $\int_{C_I} \cdots \xrightarrow{\varepsilon \to 0} \int_{-1}^{+1} \frac{dx}{(x+2)\sqrt{1-x^2}} = I$

$\int_{C_{II}} \cdots \xrightarrow{\varepsilon \to 0} -\int_{+1}^{-1} \cdots = I$

Also

$\left| \int_{C_R} \cdots \right| \le \int_0^{2\pi} \frac{R\, d\vartheta}{(R-2)(R^2-1)^{1/2}} \xrightarrow{R \to \infty} 0$

$\left| \int_{C_\varepsilon} \cdots \right| \le \left| \int_{\pi}^{-\pi} \frac{\varepsilon\, d\vartheta}{(2-\varepsilon)\, \varepsilon^{1/2} \cdot (1-\varepsilon)^{1/2}} \right| \xrightarrow{\varepsilon \to 0} 0$; same for C'_ε.

So $2I = \oint \cdots = 2\pi i \cdot \frac{1}{i\sqrt{3}} \Rightarrow I = \frac{\pi}{\sqrt{3}}$