Problem Set 6
Math. 466
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Due: [illegible]

1. (a) Given $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$, $\Gamma(\beta) = \int_0^\infty y^{\beta-1} e^{-y} dy$

show that

$$\Gamma(\alpha)\,\Gamma(\beta) = 2\Gamma(\alpha+\beta)\int_0^{\pi/2} \cos^{2\alpha-1}\vartheta \sin^{2\beta-1}\vartheta \, d\vartheta$$

$$= \Gamma(\alpha+\beta) \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$$

(b) Use the result from problem 4.3c, namely that $\int_{-\infty}^{\infty} \frac{e^{\alpha t}}{1+e^t} dt = \frac{\pi}{\sin\pi\alpha}$

to show that $\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha}$.

2. The Laplace transform of $f(t)$ is $F(s) = \frac{1}{(s-a)^{\nu+1}}$, $Re(\nu) > -1$. Find $f(t)$ by contour integration (The branch should be chosen so that $F(s)$ is real when s is real, $s>a$. Here a is a real constant).

3. By using Laplace x-forms solve the following problem for $u(t)$:

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$$\frac{du}{dt} + \int_0^t u(x)\,dx = H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad u(0) = 2$$

4. $f(t)$ is periodic with period T, i.e. $f(t+nT) = f(t)$, for n an integer.

$$g(t) = \begin{cases} f(t), & 0 < t < T \\ 0, & t > T \end{cases}$$

Show: $F(s) = \dfrac{G(s)}{1-e^{-sT}}$ for $s > 0$

Use this result to show that if $f(t) = |\sin \omega t|$, then $F(s) = \dfrac{\omega}{s^2+\omega^2} \coth\left(\dfrac{\pi s}{2\omega}\right)$.

5. Abel's integral equation for the function $\phi(t)$, $t>0$ is $\int_0^t \phi(\tau)(t-\tau)^{-\alpha}\,d\tau = f(t)$, $t>0$, where α is constant, $0<\alpha<1$, and $f(t)$ is given. Solve for $\phi(t)$ using Laplace x-forms.

6. Given that

$$J_0(at) = \frac{1}{\pi}\int_0^{\pi} \cos(at\sin\theta)\,d\theta$$

show that the Laplace transform of $J_0(at)$ is $(s^2+a^2)^{-1/2}$. Hence prove that $\int_0^t J_0(\tau)\,J_0(t-\tau)\,d\tau = \sin t$.

7. (See class notes)

Problem*: The equation

$$\frac{\partial^2 u}{\partial x^2} - k^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad (6.7)$$

where k, and c are physical constants, govern the motion of a string on an elastic foundation. Consider the case of a semi-infinite string, initially at rest with zero displacement, and subject to the boundary condition

$$u = \sin \omega t, \quad x = 0, \quad t > 0.$$

Using a Laplace transform method, find the solution in the transformed domain. Outline the main steps involved in the inversion for $\omega > kc$.