Homework II Solutions

(1a) $(z-1) = \left[\frac{1}{2}(z-1)^2\right]' \Rightarrow$

$$\int_1^i (z-1)\,dz = \frac{1}{2}(z-1)^2\Big|_1^i = \frac{1}{2}(-1+i)^2 = -i$$

(1b) $$I = \int_{-i}^{i}(z^2+iy^2)\,dz = \int_{-i}^{i} z^2\,dz + i\int_{-i}^{i} y^2\,dz = \frac{z^3}{3}\Big|_{-i}^{i} + I_1 = -\frac{2}{3}i + I_1$$

I_1: $z = e^{i\vartheta}$, $dz = ie^{i\vartheta}d\vartheta$, $y = \sin\vartheta$; $-\pi/2 \le \vartheta \le \pi/2$; $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$

$$I_1 = i\int_{\vartheta=-\pi/2}^{\pi/2} \sin^2\vartheta\, i e^{i\vartheta}\,d\vartheta = -\int_{-\pi/2}^{\pi/2} \sin^2\vartheta(\cos\vartheta + \underbrace{i\sin\vartheta)\,d\vartheta}_{=0}$$

odd function over symmetric range

$$= -\int_{-\pi/2}^{\pi/2} \frac{d}{d\vartheta}\left(\frac{1}{3}\sin^3\vartheta\right)d\vartheta = -\frac{1}{3}\sin^3\vartheta\Big|_{-\pi/2}^{\pi/2} = -\frac{2}{3}$$

$$I = -\frac{2}{3}(1+i)$$

(1c) $$\oint_{|z|=1}\left(1+\frac{2}{z}\right)dz = \underbrace{\oint dz}_{=0,\ \text{analytic}} + 2\oint\frac{dz}{z} = 2\cdot 2\pi i = 4\pi i$$

(2) $$\frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2}\left(e^{i\vartheta}+e^{-i\vartheta}\right) = \cos\vartheta\ ; \quad \frac{dz}{z} = \frac{d(e^{i\vartheta})}{e^{i\vartheta}} = i\,d\vartheta \quad \text{on } z = e^{i\vartheta}$$

so $$\frac{1}{2\pi i}\oint_c \frac{e^{t/2(z+\frac{1}{z})}\,dz}{z} = \frac{1}{2\pi}\int_{\vartheta=0}^{2\pi} e^{t\cos\vartheta}\,d\vartheta$$

On the other hand, since $e^u = 1 + u + \frac{1}{2!}u^2 + \dots + \frac{1}{n!}u^n + \dots$,

setting $u = \frac{t}{2z}$ we have $e^{t/2z} = 1 + \frac{t}{2}\frac{1}{z} + \dots + \frac{1}{n!}\left(\frac{t}{2}\right)^n\frac{1}{z^n} + \dots$

and $$\frac{1}{z}e^{t/2(z+1/z)} = e^{\frac{tz}{2}}\left(\frac{1}{z} + \frac{t}{2}\frac{1}{z^2} + \dots + \frac{1}{n!}\left(\frac{t}{2}\right)^n\frac{1}{z^{n+1}} + \dots\right)$$

Substituting: $$\frac{1}{2\pi i}\oint\frac{e^{t/2(z+\frac{1}{z})}dz}{z} = \frac{1}{2\pi i}\oint\frac{e^{tz/2}}{z}dz + \frac{1}{2\pi i}\oint\frac{t}{2}\frac{e^{tz/2}}{z^2}dz$$

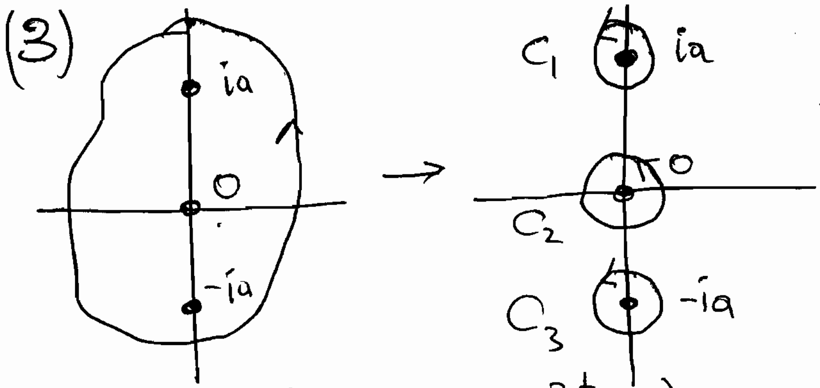
$$+ \dots + \frac{1}{n!}\left(\frac{t}{2}\right)^n\frac{1}{2\pi i}\oint\frac{e^{tz/2}}{z^{n+1}}dz + \dots$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{2}\right)^n\left(\frac{1}{2\pi i}\oint\frac{e^{tz/2}}{z^{n+1}}dz\right)$$

$$\frac{1}{n!}\frac{d^n}{dz^n}e^{tz/2}\Big|_{z=0} = \frac{1}{n!}\left(\frac{t}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)^2\left(\frac{t}{2}\right)^{2n} = I_0(t)$$

(3)



$$\frac{e^{zt}}{z^2(z^2+a^2)} = \frac{e^{zt}}{z^2(z+ia)(z-ia)}$$

$$I = \oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} = I_1 + I_2 + I_3$$

$$I_1 = \frac{k}{2\pi i}\oint_{C_1} \frac{1}{z-ia}\underbrace{\left(\frac{e^{zt}}{z^2(z+ia)}\right)}_{\text{analytic inside } C_1} dz = k\left(\frac{e^{zt}}{z^2(z+ia)}\right)_{z=ia} = \frac{ke^{iat}}{(ia)^2 \cdot 2ia}$$

Cauchy integral formula

$$= -\frac{k}{2ia^3}e^{iat}$$

similarly

$$I_3 = \frac{k}{2\pi i}\oint_{C_3} \frac{1}{z+ia}\left(\frac{e^{zt}}{z^2(z-ia)}\right)dz = \cdots \underset{(\text{at } z=-ia)}{} = \frac{k}{2ia^3}e^{-iat}$$

$$I_2 = \frac{k}{2\pi i}\oint_{C_2} \frac{1}{z^2}\left(\frac{e^{zt}}{z^2+a^2}\right)dz = k\frac{d}{dz}\left(\frac{e^{zt}}{z^2+a^2}\right)_{z=0} = \cdots = \frac{kt}{a^2}$$

$$= \left.\frac{te^{zt}}{z^2+a^2} - \frac{2ze^{zt}}{(z^2+a^2)^2}\right|_{z=0}$$

$$I = \frac{kt}{a^2} - \frac{k}{a^3}\left(\frac{e^{iat}-e^{-iat}}{2i}\right)$$

$$= \frac{kt}{a^2} - \frac{k}{a^3}\sin at$$

(i.e. angular velocity increases linearly but suffers an oscillatory modulation)

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(4) If $f(z)$ entire, then $e^{f(z)}$ entire. Now $|e^{f(z)}| = |e^u e^{iv}| = e^u$ (since $|e^{iv}|=1$ for real v).

Since $u = \mathrm{Re} f(z) < M \Rightarrow e^u < e^M \Rightarrow |e^{f(z)}|$ bounded entire function $\Rightarrow e^{f(z)}$ constant $\Rightarrow f(z)$ constant.

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(5) (a) Easy proof: $f(z)$ entire $\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n$, valid $\forall z$: On real axis, z real, f real $\Rightarrow a_n$ real.

On imaginary axis, $z = iy$, f imaginary $\Rightarrow$ no even powers of z are present (since for n even, $(iy)^n$ real)

$\Rightarrow$ only $a_n \neq 0$ for n odd $\Rightarrow f(z) = \sum_{n=1}^{\infty} a_{2n+1} z^{2n+1}$

$\Rightarrow f(-z) = -f(z)$, f odd.

(6) $f(z) = \sum_{k=0}^{\infty} \frac{k^3}{3^k} z^k = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) z^k$ (Taylor)

$\Rightarrow f^{(k)}(0) = \frac{k!\,k^3}{3^k}$; $R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim \left(\frac{k}{k+1}\right)^3 \frac{3^{k+1}}{3^k} = 3$

(a) $f^{(6)}(0) = 6!\,6^3/3^6$ $\quad (f(0) = 0)$.

(b) $\oint_{|z|=1} \frac{f(z)}{z^5} dz = \frac{2\pi i}{4!} f^{(4)}(0) = 2\pi i \cdot 4^3/3^4$

(c) $e^z f(z)$ analytic for $|z|<3 \Rightarrow \oint_{|z|=1} f(z) e^z dz = 0$ (Cauchy).

(d) $\oint \frac{f(z)\sin z}{z^2} dz = 2\pi i \frac{d}{dz}\left(f(z)\sin z\right)\Big|_{z=0}$

$= 2\pi i\left(f'(0)\sin 0 - f(0)\cos 0\right) = 2\pi i\left(\frac{1}{3}\cdot 0 - 0\cdot 1\right) = 0$

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(7) (a) $z + \frac{(a-b)z^2}{2!} + \frac{(a-b)(a-2b)z^2}{3!} z^3 + \dots = \sum_{k=1}^{\infty} a_k z^k$

$a_1 = 1,\ a_n = \frac{1}{n!} \prod_{k=1}^{n-1} (a-kb)$

$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{1}{n!}(a-b)\cdots(a-(n-1)b) \Big/ \frac{1}{(n+1)!}(a-b)\cdots(a-(n-1)b)(a-nb)\right|$

$= \left|\frac{n+1}{a-nb}\right| \xrightarrow{n\to\infty} \frac{1}{b} = R$

(b) $\left|\frac{a_n}{a_{n+1}}\right| = \frac{n}{2^n} \Big/ \frac{n+1}{2^{n+1}} = \frac{2^{n+1}}{2^n}\cdot\frac{n}{n+1} = \frac{2n}{n+1} \xrightarrow{n\to\infty} 2 = R$

(c) $\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n+2} \Big/ \frac{n+2}{n+3} = \frac{(n+1)(n+3)}{(n+2)(n+2)} \xrightarrow{n\to\infty} 1$

(8) $\frac{P'}{P} = \frac{(z-z_1)'}{z-z_1} + \dots + \frac{(z-z_n)'}{z-z_n} = \sum_{k=1} \frac{1}{z-z_k}$

$$I = \frac{1}{2\pi i} \oint \frac{P'(z)}{P(z)} dz = \sum_{k=1}^{n} \frac{1}{2\pi i} \oint \frac{dz}{z-z_k}$$

But $\oint_C \frac{dz}{z-z_k} = \begin{cases} 2\pi i, & z_k \text{ inside } C \\ 0 & z_k \text{ outside} \end{cases}$

So $I = N = \#$ of roots of $P(z)$ inside C.

(5b) (Hard but instructive proof): $f(z) = u(z) + i v(z)$

We write $z = x+iy$, $u(z) = u(x,y)$, $v(z) = v(x,y)$ where the first argument is the real, the second the imaginary part of z. $z^* = x - iy$

Now $f^*(z) = u(x,y) - i v(x,y)$ $\quad f(z^*) = u(x,-y) + i v(x,-y)$

$f(iz) = f(-y+ix) = u(-y,x) + i v(-y,x)$, $f^*(iz) = u(-y,x) - i v(-y,x)$

$i f(iz) = -v(-y,x) + i u(-y,x)$, $i f(iz^*) = -v(y,x) + i u(y,x)$

$i f^*(iz) = v(-y,x) + i u(-y,x)$.

1) f real on real axis: (Schwarz): $f^*(z) = f(z^*)$

$\Rightarrow u(x,y) - i v(x,y) = u(x,-y) + i v(x,-y)$

$\Rightarrow u(x,y) = u(x,-y)$ (1a), $v(x,y) = -v(x,-y)$ (1b)

2) let $g(z) = i f(iz)$; since $f(z)$ imaginary on imaginary axis

$\Rightarrow f(iy) = u(0,y) + i v(0,y) = i v(0,y)$, i.e. $u(0,y) = 0$

Then $g(x) = i f(ix) = -v(0,x) + i u(0,x) = -v(0,x)$ real

i.e. $g(z)$ real on real axis $\Rightarrow g^*(z) = g(z^*) \Rightarrow$

$-i f^*(iz) = i f(iz^*) \Rightarrow -v(-y,x) - i u(-y,x) = -v(y,x) + i u(y,x)$

$\Rightarrow -v(-y,x) = -v(y,x) \Rightarrow v(x,y) = v(-x,y)$ (2b) (interchange variables – meaning unaffected)

$-u(-y,x) = u(y,x) \Rightarrow u(x,y) = -u(-x,y)$ (2a)

Combine (1ab), (2ab) to get $f(-z) = -f(z)$