Homework III Solutions

(1) $P = \sum_0^\infty a_k z^k$; $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{6}$ ..

$$P^{-1} = \sum_0^\infty b_n z^n \equiv \sum_0^\infty \frac{B_n}{n!} z^n$$

$$b_k = \frac{(-1)^k}{a_0^{k+1}} \det \begin{pmatrix} a_1 & a_2 \cdots & a_k \\ a_0 & \ddots & \vdots \\ & \ddots & a_2 \\ 0 & a_0 & a_1 \end{pmatrix} \; ; \; b_0 = 1, \quad b_1 = -1 \left| \tfrac{1}{2} \right| = -\tfrac{1}{2}$$

$$b_2 = 1 \cdot \begin{vmatrix} 1/2 & 1/6 \\ 1 & 1/2 \end{vmatrix} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}, \quad b_3 = -\begin{vmatrix} 1/2 & 1/6 & 1/24 \\ 1 & 1/2 & 1/6 \\ 0 & 1 & 1/2 \end{vmatrix} = 0$$

$$b_4 = \begin{vmatrix} 1/2 & 1/6 & 1/24 & 1/120 \\ 1 & 1/2 & 1/6 & 1/24 \\ 0 & 1 & 1/2 & 1/6 \\ 0 & 0 & 1 & 1/2 \end{vmatrix} = \frac{1}{2} \cdot 0 - 1 \cdot \left(\frac{1}{6} \cdot \frac{1}{24} - \begin{vmatrix} 1/2 & 1/120 \\ 1 & 1/24 \end{vmatrix} \right)$$

$$= -1/30 \cdot 24$$

$\Rightarrow B_0 = 1$, $B_1 = -1/2$, $B_2 = 2 \cdot 1/12 = 1/6$, $B_3 = 0$, $B_4 = -1/30$ ...

(A computation based on the Wronski formula is rather awkward here; it is better to use the Cauchy product formula directly : $\sum_{k=0}^{n} a_k b_{n-k} = 0$, $n = 1, 2, \ldots$: $b_n = -\frac{1}{a_0} \sum_{k=0}^{n-1} a_k b_{n-k}$

with $b_0 = 1/a_0 = 1$).

(2) (i) $a_{n+2} = a_{n+1} + 2a_n$: 2^{nd} order difference equation.

let $a_n = r^n \Rightarrow r^{n+2} = r^{n+1} + 2r^n \Rightarrow r^2 - r - 2 = 0$; $r = -1, 2$.

General solution: $a_n = C_1 (-1)^n + C_2 2^n$

$$\left. \begin{aligned} a_0 &= C_1 + C_2 = 1 \\ a_1 &= -C_1 + 2C_2 = 1 \end{aligned} \right\} \quad \begin{aligned} C_2 &= 2/3 \\ C_1 &= 1/3 \end{aligned}$$

$\therefore a_n = \frac{1}{3}(-1)^n + \frac{2}{3} 2^n$ $\quad (a_2 = 3,\ a_3 = 5,\ a_4 = 11, \text{etc.})$

(ii) $A^{-1} = \sum b_k z^k$

$$b_k = \frac{(-1)^k}{1^{k+1}} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ & 0 & \ddots & \ddots & \ddots \end{vmatrix} \qquad b_0 = 1, \quad b_1 = -1$$

$$b_2 = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2$$

Clearly, once 3 or more rows are included, the determinant vanishes because (row 2) + 2·(row 3) = (row 1) because of the recurrence relation $a_{n+1} = a_n + 2a_{n-1}$.

So $\quad A^{-1}(z) = 1 - z - 2z^2$.

Note: since $1 - z - 2z^2 = \cancel{(z+1)(z+2)} = (z+1)(1-2z)$, the series $a_0 + a_1 z + a_2 z^2 + \cdots = \dfrac{1}{1 - z - 2z^2}$ has radius of convergence $R = 1/2$ (distance from $z=0$ to $z = 1/2$ the nearest pole).

(3) $\displaystyle Q_\alpha(z) = \sum_0^\infty a_n z^n = (1+z)^\alpha \; ; \quad a_n = \binom{\alpha}{n}$

Recall that $\displaystyle \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ with $\displaystyle \binom{\alpha}{0} := 1$.

Now, since $(1+z)^\alpha (1+z)^\beta = (1+z)^{\alpha+\beta}$, we have that

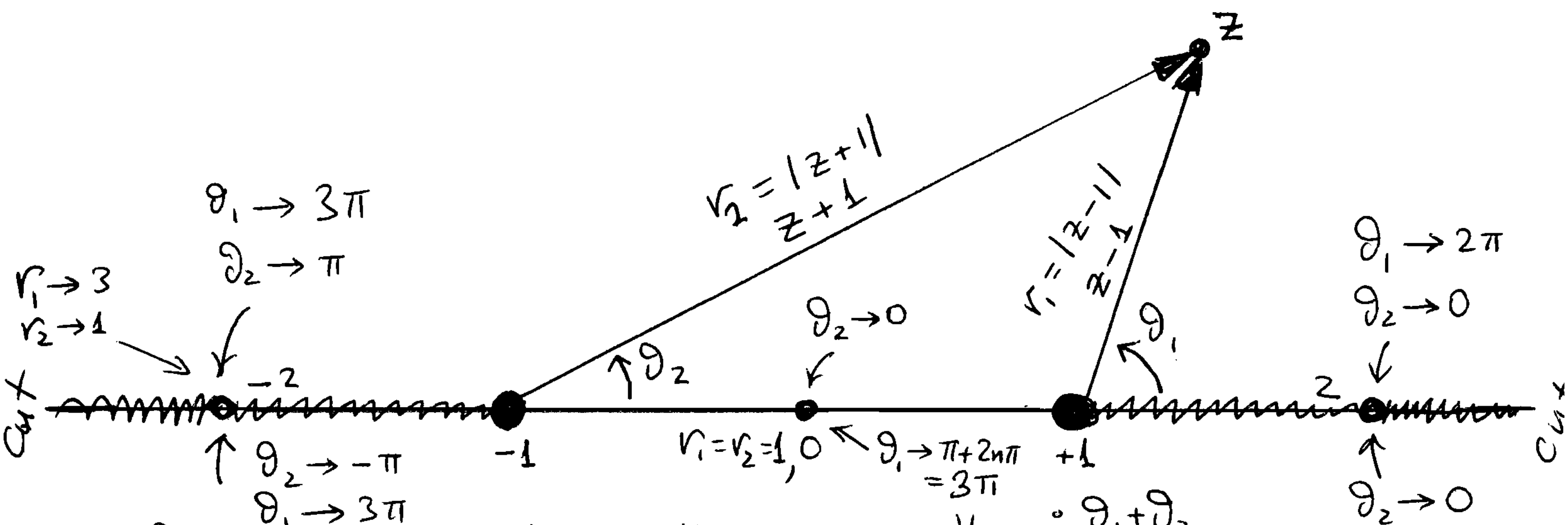
$$\underset{\sum_0^\infty a_n z^n}{Q_\alpha} \; \underset{\sum_0^\infty b_n z^n}{Q_\beta} = \underset{\sum_0^\infty c_n z^n}{Q_{\alpha+\beta}} \quad \Rightarrow \quad \sum_0^n a_k b_{n-k} = c_n$$

$$\Rightarrow \sum_{k=0}^n \binom{\alpha}{k}\binom{\alpha}{n-k} = \binom{\alpha}{n}$$

This is an important combinatorial identity. (Vandermonde formula).

Incidentally, the radius of convergence of $Q_\alpha(z)$ is 1, the distance from 0 to $z=-1$, which is a branch point if $\alpha \neq$ positive integer. If $\alpha =$ positive integer, $\binom{\alpha}{\alpha+1} = 0$ and series terminates. ($(1+z)^n$ is a polynomial).

(8) $(1-z^2)^{1/2} = i\,(z^2-1)^{1/2}$



$$f(z) = i\,(z-1)^{1/2}(z+1)^{1/2} = i\,(r_1 r_2)^{1/2} e^{i\frac{\vartheta_1+\vartheta_2}{2}}$$

let $-\pi < \vartheta_2 < \pi$, $\quad 2n\pi < \vartheta_1 < 2n\pi + 2\pi$

We choose n to get $f(0) = 1$. Indeed

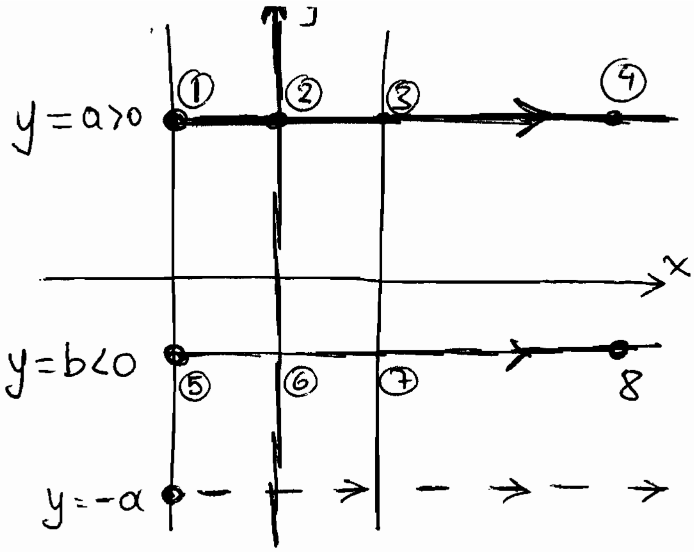
$$f(0) = i(r_1 r_2)^{1/2} e^{i\left(\frac{\vartheta_1+\vartheta_2}{2}\right)} = i\cdot 1\cdot e^{i\left(\frac{2n\pi+\pi+0}{2}\right)}$$

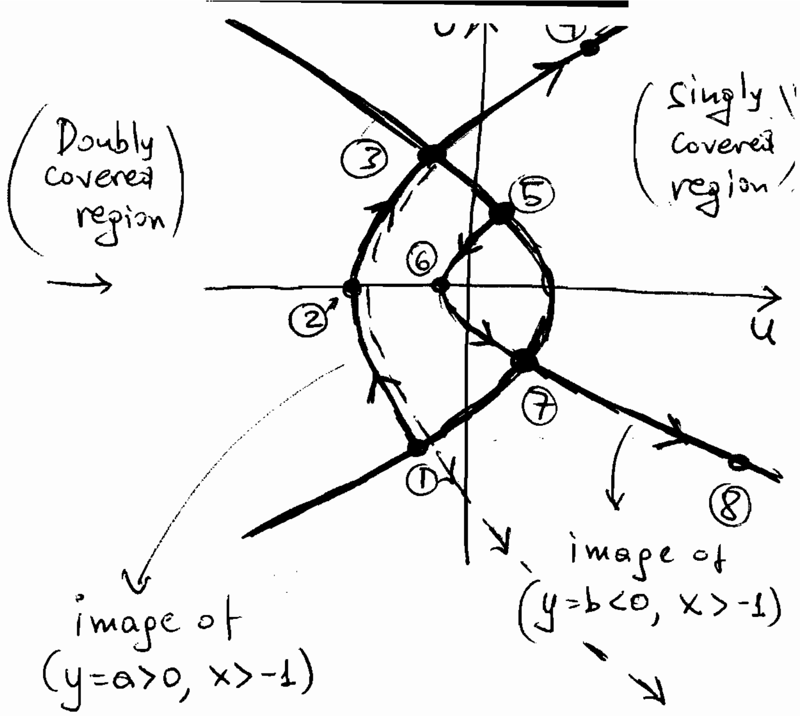
$$= i\,e^{i\frac{\pi}{2}}\,e^{in\pi} = -1\cdot(-1)^n$$

Since we are given $f(0)=1$, choice $n=1$ works

i.e. at $z=0$, $\vartheta_1 = 3\pi$. So on

	$z \to +2 \downarrow$	$z \to +2 \uparrow$	$z \to -2 \downarrow$	$z \to -2 \uparrow$
$r_1 \to$	1	1	3	3
$r_2 \to$	3	3	1	1
$\vartheta_1 \to$	2π	4π	3π	3π
$\vartheta_2 \to$	0	0	π	$-\pi$
$f(z) \to$	$i\sqrt{3}\,e^{i\frac{2\pi}{2}} = -i\sqrt{3}$	$+i\sqrt{3}$	$i\sqrt{3}\,e^{i\,4\pi/2} = i\sqrt{3}$	$i\sqrt{3}\,e^{i\frac{3\pi-\pi}{2}} = -i\sqrt{3}$





$y=a,\ x>-1:\ (a>0)$

$(u=x^2-a^2,\ v=2ax)$

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(7) $w=\sqrt{z} \Rightarrow u^2-v^2+2uvi = x+iy$, $\underline{x\geq 1}$

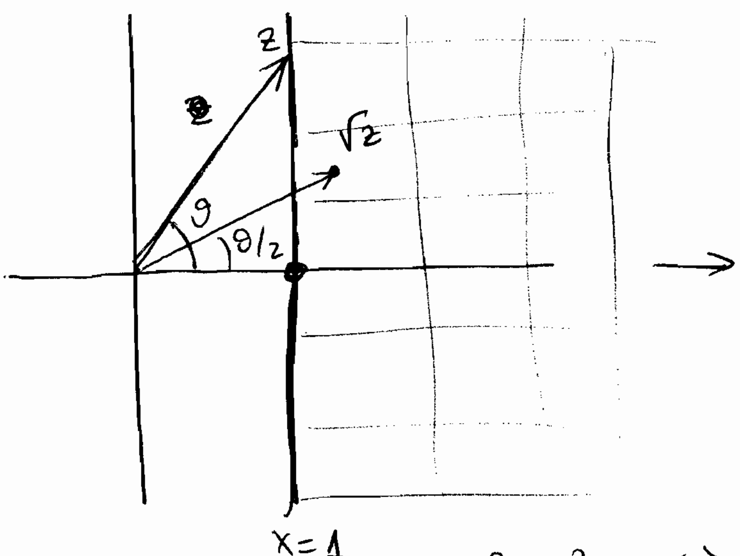
Since $\sqrt{1}=1 \Rightarrow u^2-v^2=1,\ 2uv=0 \Rightarrow \underline{v=0,\ u^2=1} \Rightarrow u=\pm 1$

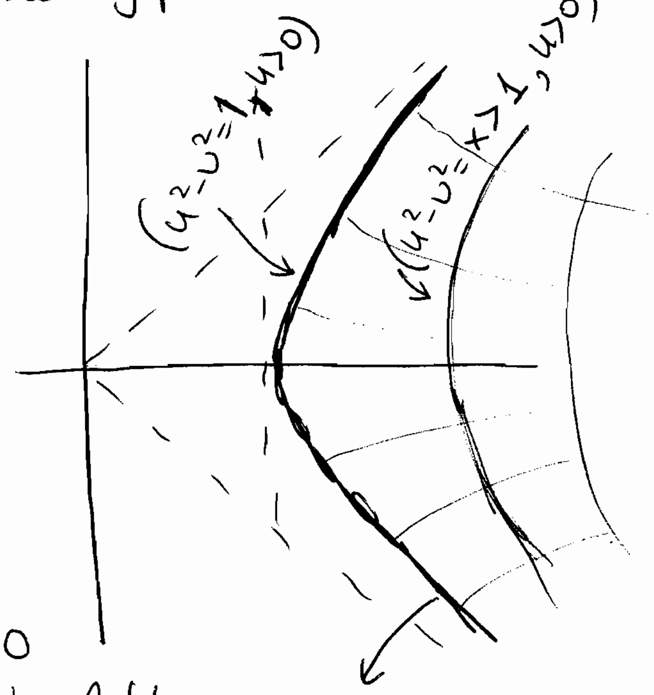
leads to the single choice $u=1$. Then the imag

of the line $x=1$ is:

$u^2-v^2+2iuv = 1+iy \Rightarrow \left.\begin{array}{l} u^2-v^2=1 \\ 2uv=y \end{array}\right\}$ since $\left\{\begin{array}{l} v=0 \\ \Rightarrow u=+1 \end{array}\right\}$

we keep only one branch of the hyperbola.





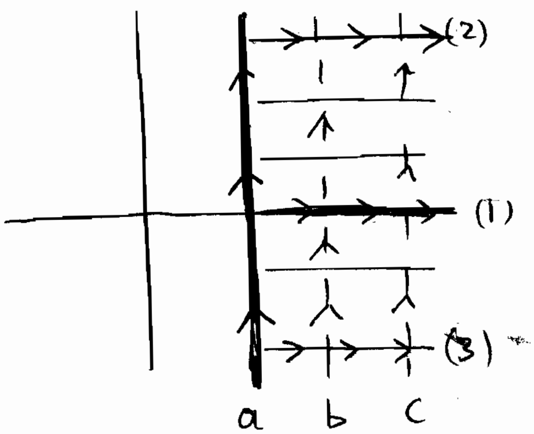
and for $x>1$: $u^2-v^2=x>1,\ u>0$

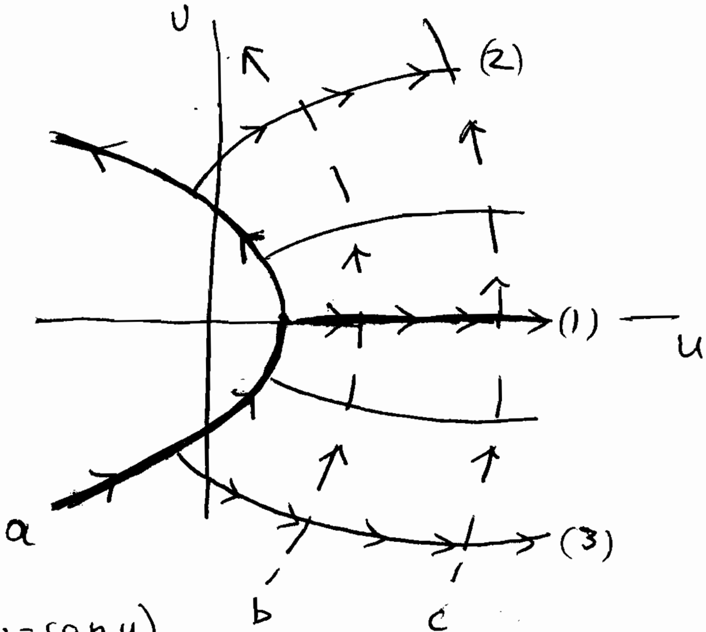
gives hyperbolas to the left

of $(u^2-v^2=1,\ u>0)$

(6) $w = z^2 = x^2 - y^2 + i(2xy) \quad (u = x^2 - y^2,\ v = 2xy)$.

(a) Image of $x > 1$:

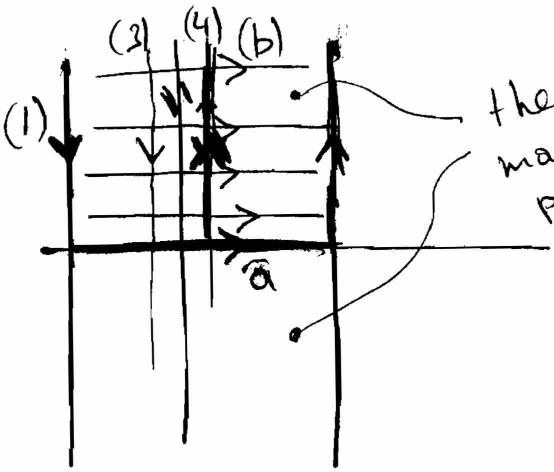




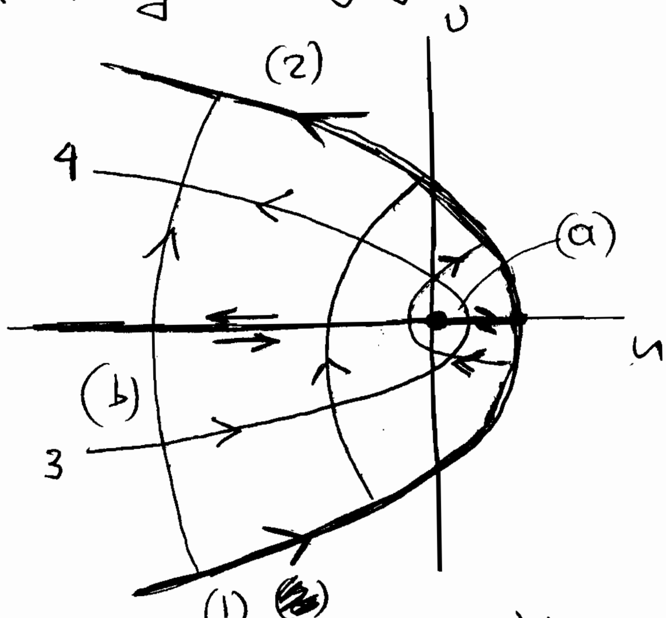
$x = 1: \ (u = 1 - y^2,\ v = 2y) \Rightarrow u = 1 - \frac{1}{4}v^2; \ (\operatorname{sgn} v = \operatorname{sgn} y)$

$x = a: \ (u = a^2 - y^2,\ v = 2ay) \Rightarrow u = a^2 - \frac{1}{4a^2}v^2$

$x = -1: \ (u = 1 - y^2,\ v = -2y) \Rightarrow u = 1 - \frac{1}{4}v^2 \ (\text{but } \operatorname{sgn} v = -\operatorname{sgn} y)$



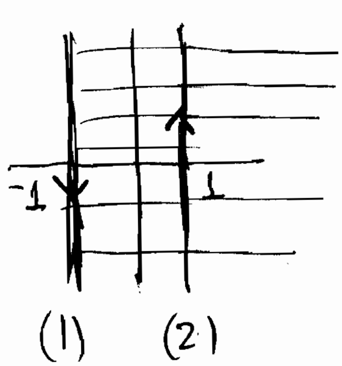
these strips map to the same place, but "reversed"!

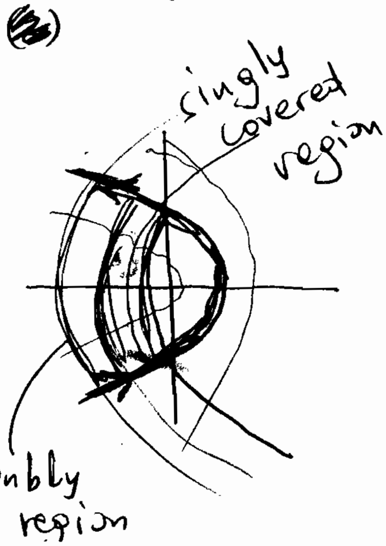


$y = 0: \ -1 \le x \le 1 \ \Big\} \ u = x^2,\ v = 0$

$y = a > 0, \ -1 \le x \le 1 \ \Big\} \ u = x^2 - a^2, \ v = 2a$

$x = 0, \ y > 0 \ \Big\} \ u = -y^2, \ v = 0$





(4) Clearly, $0<|z|<1$:

$$-\left(\frac{1}{z}+1+z+z^2+\dots\right) = -\frac{1}{z}\left(1+z+z^2+\dots\right) = -\frac{1}{z(1-z)} \equiv f_1(z)$$

while, if $|z|>1 \Rightarrow |1/z|<1$:

$$\frac{1}{z^2}+\frac{1}{z^3}+\dots = \frac{1}{z^2}\left(1+\frac{1}{z}+\dots\right) = \frac{1}{z^2}\,\frac{1}{1-1/z} = \frac{1}{z(z-1)} \equiv f_2(z).$$

Clearly, if we define $f(z) = \frac{1}{z(z-1)}$, $z\neq 0,1$
it is a function which is analytic in both domains.

$f(z)$ is analytic for $0<|z|<1$, where $f(z) \equiv f_1(z)$,
$f(z)$ " " " $|z|>1$, " $f(z) \equiv f_2(z)$.

In this sense, $f(z)$ constitutes the analytic continuation of both series to $\mathbb{C}\setminus\{0,1\}$ (i.e. the entire complex plane minus the point $0,1$).

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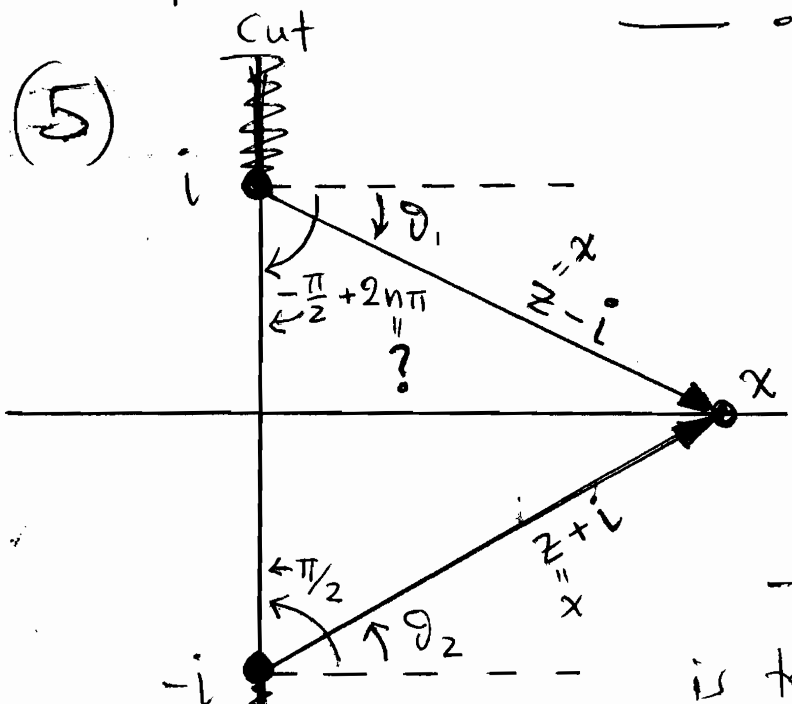
(5)



Preliminaries

Cut plane as shown:

$$z-i = |z-i|\,e^{i\vartheta_1}$$
$$\frac{\pi}{2}+2n\pi < \vartheta_1 < -\frac{3\pi}{2}+2n\pi$$

$$z+i = |z+i|\,e^{i\vartheta_2}$$
$$-\frac{\pi}{2} < \vartheta_2 < \frac{3\pi}{2}$$

The point of this problem is to establish that

$$\arctan z = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right)$$ gives

the correct value of $\arctan x$ when $z=x$.

Cut

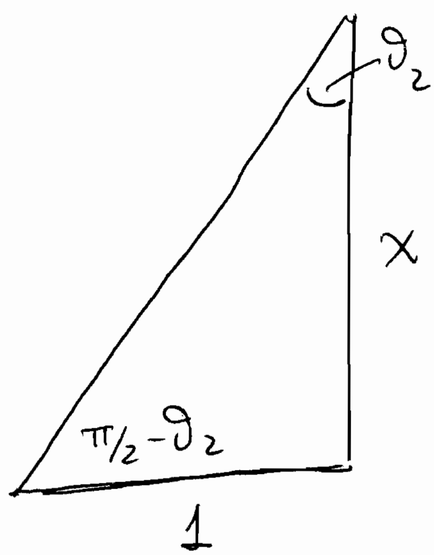
(5) (continued) We interpret the formula for arctan z when $z = x$ real: let $f(z) = \arctan z$. Then

$$f(z) = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right) = \frac{1}{2i}\log\left(\left(\frac{r_1}{r_2}\right)e^{i(\vartheta_1-\vartheta_2+\pi)}\right)$$

$$\left.\begin{aligned}(z-i) &= |z-i|e^{i\vartheta_1} \equiv r_1 e^{i\vartheta_1}\\ (z+i) &= |z+i|e^{i\vartheta_2} \equiv r_2 e^{i\vartheta_2}\end{aligned}\right\} \quad = \frac{1}{2i}\left\{\log\left(\frac{r_1}{r_2}\right) + i\left(\vartheta_1-\vartheta_2+\pi\right)\right\}$$

$$\Rightarrow f(z) = \frac{1}{2i}\log\left(r_1/r_2\right) + \frac{\vartheta_1-\vartheta_2+\pi}{2}$$

when $z = x + i0$ (real) then $r_1 = r_2 = \sqrt{x^2+1}$, so $\log(r_1/r_2) = \log 1 = 0$ and $\vartheta_1 = -\vartheta_2$ so $f(x) = \frac{\pi}{2} - \vartheta_2$ (with ϑ_2 positive). But look at triangle:



$$\tan\left(\frac{\pi}{2}-\vartheta_2\right) = \frac{x}{1} = x$$

$$\Rightarrow \arctan x = \frac{\pi}{2} - \vartheta_2$$

So, indeed, complex formula gives the correct value for arctan x when $z \to x$, so it constitutes the analytic continuation of arctan x to complex values.

(9) $$w = \frac{z-i}{z+i} = \frac{x+i(y-1)}{x+i(y+1)} \cdot \frac{x-i(y+1)}{x-i(y+1)}$$

$$= \frac{x^2+(y-1)(y+1)+i\left[x(y-1)-x(y+1)\right]}{x^2+(y+1)^2}$$

$$= \frac{x^2+y^2-1 \; - 2xi}{x^2+(y+1)^2} = u+iv$$

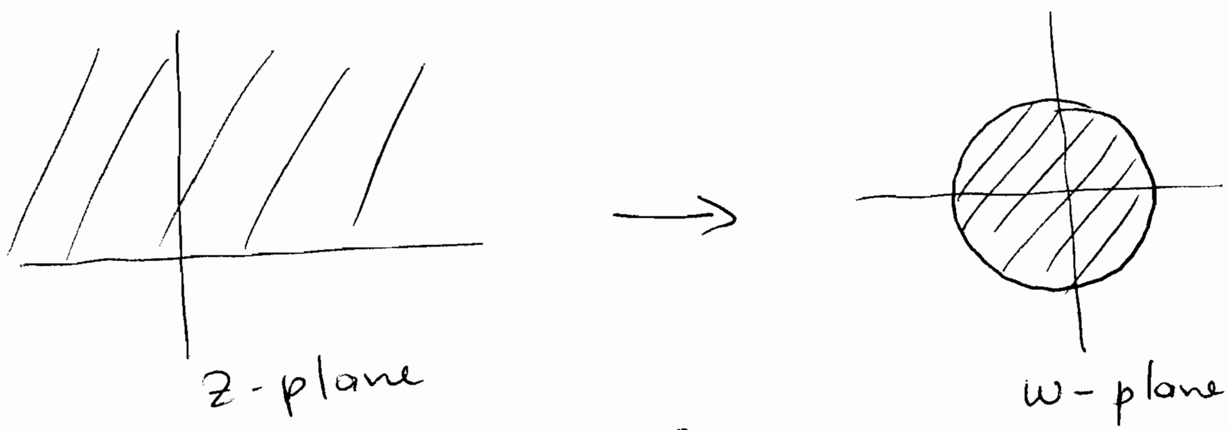
$$\Rightarrow u = \frac{x^2+y^2-1}{x^2+(y+1)^2}, \quad v = \frac{-2x}{x^2+(y+1)^2}$$

The interior of the unit circle on the (u,v) plane is given by $u^2+v^2<1 \Rightarrow$ substituting

$$\frac{(x^2+y^2-1)^2}{[x^2+(y+1)^2]^2} + \frac{4x^2}{[x^2+(y+1)^2]^2} < 1 \Rightarrow \text{(after algebra and cancellations)}$$

always positive

$$0 < 4y\left[(y+1)^2+x^2\right] \Rightarrow y>0$$



$$w = \frac{z-i}{z+i}$$