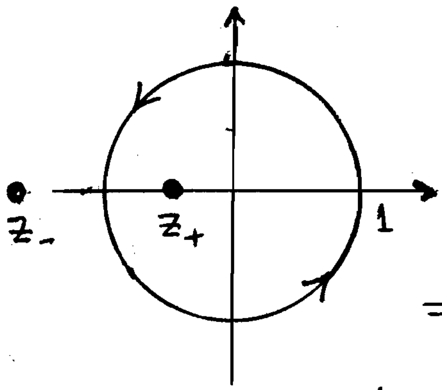
Solutions Set V

(1) $I = \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}$

since $\cos(\pi+\theta) = \cos(\pi-\theta)$.

Let $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z+1/z)$. Then

$$2I = \oint_{|z|=1} \frac{dz}{iz\left(a+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2} = -4i \oint \frac{z\,dz}{\underbrace{(z^2+2az+1)^2}_{=(z-z_+)^2(z-z_-)^2}}$$



$z_\pm = -a \pm \sqrt{a^2-1}$: (double poles)

Since $z_+ z_- = 1$ and $|z_-| > 1 \Rightarrow |z_+| < 1$: inside

So $I = -2i \cdot 2\pi i \cdot \mathrm{Res}(z_+)$

$$\Rightarrow I = 4\pi \cdot \lim_{z\to z_+} \frac{d}{dz} \cancel{(z-z_+)^2} \cdot \frac{z}{\cancel{(z-z_+)^2}(z-z_-)^2}$$

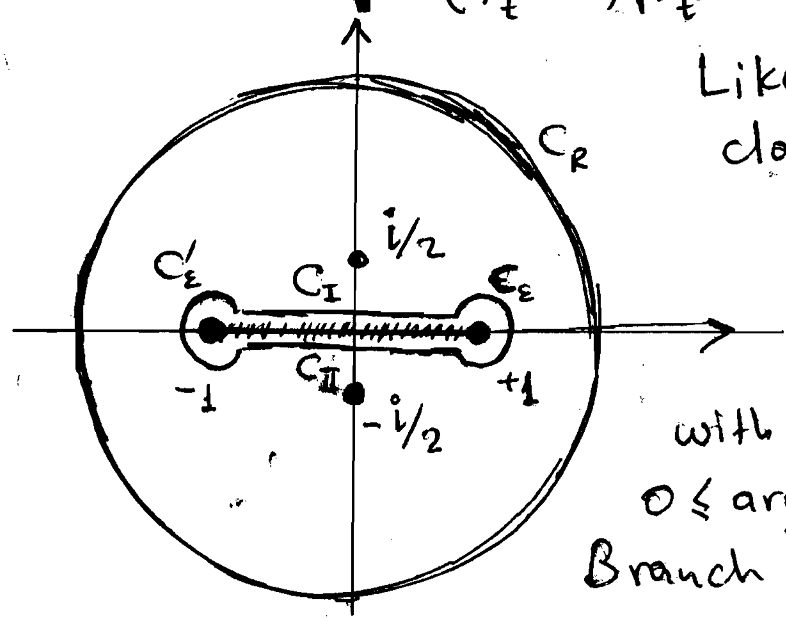
$$\Rightarrow I = 4\pi \left[\frac{1}{(z-z_-)^2} - \frac{2z}{(z-z_-)^3}\right]_{z=z_+} = -4\pi \left.\frac{z+z_-}{(z-z_-)^3}\right|_{z=z_+} =$$

$$= -4\pi \frac{z_+ + z_-}{(z_+ - z_-)^3} = -4\pi \cdot \frac{-2a}{8(a^2-1)^{3/2}} \Rightarrow I = \frac{\pi a}{(a^2-1)^{3/2}}$$

(2.) let $x = 1/t$, $dx = -\frac{dt}{t^2}$, $\left.\begin{array}{l} x=1 \Rightarrow t=1 \\ x=\infty \Rightarrow t=0 \end{array}\right\}$

$$I = \int_1^0 \frac{1/t \cdot \left(-dt/t^2\right)}{\left(1/t^2+4\right)\sqrt{1/t^2 - 1}} = \int_0^1 \frac{dt}{(1+4t^2)\sqrt{1-t^2}} = \frac{1}{2}\int_{-1}^{1} \text{''}$$

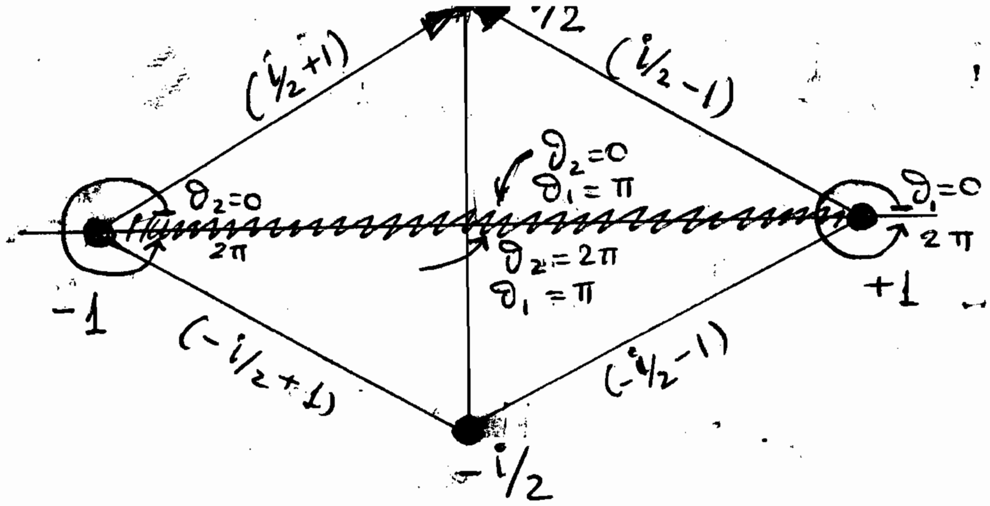
by symmetry



Like we did in the example in class, we take

$$\sqrt{1-z^2} = -i\sqrt{z^2-1}$$
$$= -i(r_1 r_2)^{1/2} e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}$$

with $r_1 = |z-1|$, $r_2 = |z-2|$ and $0 \leq \arg(z-1) \equiv \theta_1$, $\arg(z+1) = \theta_2 < 2\pi$. Branch cut lies between $+1$ & -1.



Definitions for $\sqrt{z^2-1}$.

We consider $B = i\oint_C \frac{dz}{(1+4z^2)\sqrt{z^2-1}}$, $C = C_R + C_\varepsilon + C_\varepsilon' + C_I + C_{II}$

Clearly $B = 2\pi i \sum \text{Res}$, with simple poles at $\pm i/2$.

First we verify that the above contour integral leads to the desired integral.

(i) On C_I: $\int_{C_I} = i\int_{-\varepsilon}^{\varepsilon} \frac{dx}{(1+4x^2)\, i\sqrt{1-x^2}} \xrightarrow{\varepsilon\to 0} I$

$\Big(z = x$

$(z-1) = |x-1|e^{i\pi}$, $z+1 = |x+1|e^{i0}$,

$\sqrt{z^2-1} = |x-1|^{1/2}|x+1|^{1/2}e^{i\left(\frac{\pi+0}{2}\right)} = i\sqrt{1-x^2} \Big)$

(ii) On C_{II}: $i\int_{\varepsilon}^{-\varepsilon} \frac{dx}{(1+4x^2)\cdot(-i)\sqrt{1-x^2}} \xrightarrow{\varepsilon\to 0} I$

$z = x$

$(z-1) = |x-1|e^{i\pi}$, $(z+1) = |x+1|e^{i2\pi}$

$\sqrt{z^2-1} = (|x^2-1|)^{1/2} e^{i\frac{3\pi}{2}} = -i\sqrt{1-x^2}$

So $\int_{C_I} + \int_{C_{II}} \xrightarrow{\varepsilon\to 0} 2I$, while (like in the class example) $\int_{C_\varepsilon}, \int_{C_\varepsilon'}, \int_{C_R} \longrightarrow 0$ as $R\to\infty$, $\varepsilon\to 0$.

So $\quad 2I = \oint_c \frac{i\,dz}{(1+4z^2)\sqrt{z^2-1}} = 2\pi i \cdot \left(\mathrm{Res}\left(+\frac{i}{2}\right) + \mathrm{Res}\left(-\frac{i}{2}\right)\right)$

At $(+i/2)$: $\mathrm{Res} = \lim_{z \to i/2} \frac{i}{4} \cdot \frac{z - i/2}{(z+i/2)(z-i/2)\sqrt{z^2-1}}$

$(1+4z^2) = 4(z+i/2)(z-i/2)$

$$= \frac{\cancel{i}}{4} \frac{1}{\cancel{i}\sqrt{(i/2)^2-1}}$$

Now $\sqrt{(i/2)^2-1} = |i/2 - 1|^{1/2}\,|i/2+1|^{1/2}\, e^{i\left(\frac{\vartheta_1+\vartheta_2}{2}\right)}$

from the figure, $\vartheta_1+\vartheta_2 = \pi$, $\left|\frac{i}{2} \pm 1\right| = \sqrt{1+1/4} = \frac{\sqrt{5}}{2}$

so $\sqrt{(i/2)^2-1} = \frac{\sqrt{5}}{2} e^{i\pi/2} = i\sqrt{5}/2$

~~Similarly, at z = i/2~~, So, $\mathrm{Res}(i/2) = -\frac{i}{2\sqrt{5}}$

Similarly, at $z = -i/2$,

$\mathrm{Res} = \lim_{z \to -i/2} \frac{i}{4} \frac{\cancel{z+i/2}}{\cancel{(z+i/2)}(z-i/2)\sqrt{z^2-1}} = -\frac{1}{4} \frac{1}{\sqrt{(-i/2)^2-1}}$

with $\sqrt{(-i/2)^2-1} = |(-i/2)-1|^{1/2}\,|(-i/2)+1|^{1/2}\, e^{i\left(\frac{\vartheta_1+\vartheta_2}{2}\right)}$; $|-i/2 \pm 1| = \frac{\sqrt{5}}{2}$

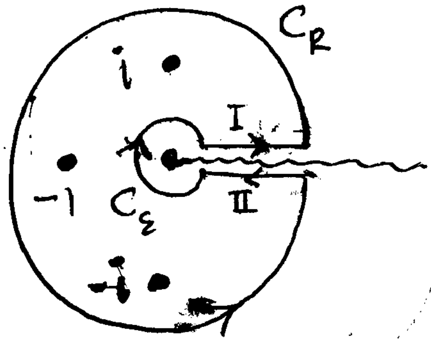
now $\vartheta_1+\vartheta_2 = 3\pi$, so $\sqrt{(-i/2)^2-1} = \frac{\sqrt{5}}{2} e^{i\frac{3\pi}{2}} = -i\sqrt{5}/2$

and $\mathrm{Res}(-i/2) = -\frac{1}{4} \frac{1}{-i\sqrt{5}/2} = -\frac{i}{2\sqrt{5}}$

Finally $\quad 2I = 2\pi i \cdot \left(-\frac{i}{2\sqrt{5}} - \frac{i}{2\sqrt{5}}\right)$

$$\Rightarrow \boxed{I = \frac{2\pi}{\sqrt{5}}}$$

(3) (a) Consider $\oint_C \frac{\ln z\, dz}{(z+1)(z^2+1)} = \int_{C_I} + \int_{C_{II}} + \int_{C_\varepsilon} + \int_{C_R}$



From the notes, $\int_{C_\varepsilon}, \int_{C_R} \to 0$ as $\substack{R \\ \varepsilon \to 0}$

So $\oint_C \longrightarrow \int_{C_I} + \int_{C_{II}} = 2\pi i \sum \text{Res}.$

We have $\ln z = \ln r + i\theta$, $0 \le \theta < 2\pi$

On C_I, $\ln z = \ln r$, $\int_{C_I} \to \int_0^\infty \frac{\ln r\, dr}{(r+1)(r^2+1)}$

On C_{II}, $\ln z = \ln r + 2\pi i$, $\int_{C_{II}} \to \int_\infty^0 \frac{(\ln r + 2\pi i)\, dr}{(r+1)(r^2+1)}$

So $\int_{C_I} + \int_{C_{II}} = -2\pi i \int_0^\infty \frac{dr}{(r+1)(r^2+1)} = -2\pi i\, I$.

The residue calculations:

$z=-1$: $\left.\frac{\ln z}{z^2+1}\right|_{z=-1} = \frac{\ln 1 + i\pi}{2} = \frac{i\pi}{2}$

$z = e^{i\pi}$

$z = i$, $z = e^{i\pi/2}$: $\left.\frac{\ln z}{(z+1)(z+i)}\right|_{z=i} = \frac{i\pi/2}{(i+1)\,2i} = \frac{\pi/4}{1+i}\cdot\frac{1-i}{1-i} = \frac{\pi}{8}(1-i)$

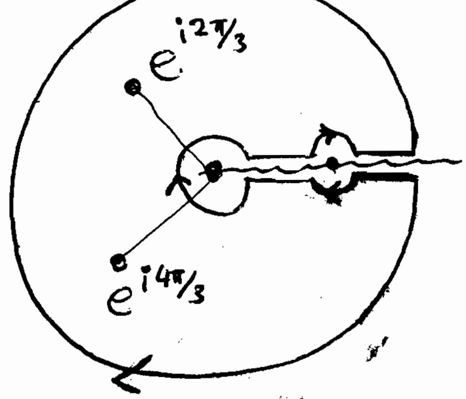
$z = -i = e^{3i\pi/2}$; $\left.\frac{\ln z}{(z+1)(z-i)}\right|_{z=-i} = \frac{3i\pi/2}{(1-i)\cdot(-2i)} = -\frac{3\pi}{8}(1+i)$

Adding: $-2\pi i\, I = 2\pi i\left[\frac{\pi i}{2} + \frac{\pi}{8}(1-i) - \frac{3\pi}{8}(1+i)\right]$

$\Rightarrow I = \pi/4$

(3b) Now, as before,

$$\oint \frac{\ln z\, dz}{z^3-1} = -2\pi i\, I$$



However, I here is a Principal Value integral and we must take special care when evaluating the residue at 1.

Indeed: $\text{Res}(1) = \frac{1}{2}(\text{res on top}) + \frac{1}{2}(\text{res. on bottom})$, refering to the branch cut on the (+) real axis:

$$\text{Res (top)} = \lim_{\substack{z\to 1 \\ \text{(from above)} \\ z \to e^{i0}}} \frac{(z-1)\ln z}{z^3-1} = \frac{1}{2z^2}\Big|_{z=1} \cdot (i0) = 0$$

$$\text{Res (bottom)} = \lim_{\substack{z\to 1 \\ \text{(from below)} \\ z\to e^{i2\pi}}} \frac{(z-1)\ln z}{z^3-1} = \frac{1}{z} \cdot \ln(e^{i2\pi}) = \pi i$$

$$\Rightarrow \text{Res}(1) = \frac{\pi i}{2}$$

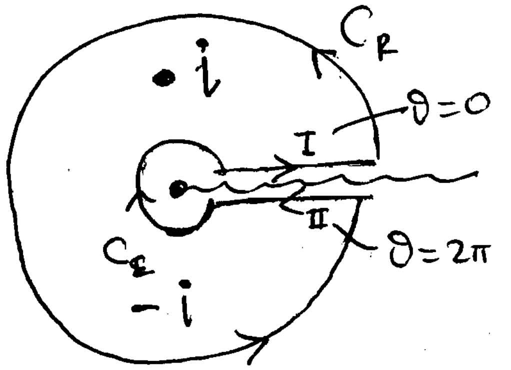
At $z = e^{2\pi i/3}$, $\text{Res} = \frac{1}{2z^2}\Big|_{z=e^{2\pi i/3}} \cdot \frac{2\pi i}{3} = \frac{\pi i}{3} e^{-4\pi i/3}$

At $z = e^{4\pi i/3}$, $\text{Res} = \frac{1}{2z^2}\Big|_{z=e^{4\pi i/3}} \cdot \frac{4\pi i}{3} = \frac{2\pi i}{3} e^{-8\pi i/3}$

Finally, $-2\pi i\, I = 2\pi i \cdot \left\{ \frac{\pi i}{2} + \frac{\pi i}{3} \underbrace{e^{-4\pi i/3}}_{-\frac{1}{2}+i\frac{\sqrt{3}}{2}} + \frac{2\pi i}{3} \underbrace{e^{-2\pi i/3}}_{-\frac{1}{2}-i\frac{\sqrt{3}}{2}} \right\}$

$$\Rightarrow I = -\pi\sqrt{3}/6$$

4. $$\oint_C \frac{\sqrt{z}\,dz}{z^2+1} = \int_I + \int_{II} + \underbrace{\int_{C_\varepsilon}}_{\to 0} + \underbrace{\int_{C_R}}_{\to 0}$$



$$\int_I \to \int_0^\infty \frac{\sqrt{x}\,dx}{x^2+1} \quad \text{as} \quad \varepsilon\to 0,\ R\to\infty$$

$$\int_{II} \to \int_\infty^0 \frac{\sqrt{x}\,\overbrace{e^{i\frac{2\pi}{2}}}^{=-1}\,dx}{x^2+1} = \int_0^\infty \frac{\sqrt{x}\,dx}{x^2+1}$$

So $$2I = \oint_C \frac{\sqrt{z}\,dz}{z^2+1} = 2\pi i\left(\mathrm{Res}(i) + \mathrm{Res}(-i)\right)$$

At $z = i = e^{i\pi/2}$, $\sqrt{z} = e^{i\pi/4}$;

$$\mathrm{Res} = \lim_{z\to e^{i\pi/2}} \frac{\sqrt{z}}{z+i} = \frac{e^{i\pi/4}}{2i} = \frac{i+1}{2\sqrt{2}\,i} = \frac{1-i}{2\sqrt{2}}$$

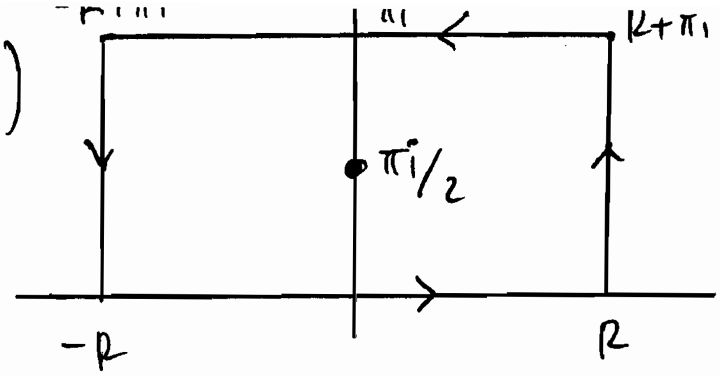
At $z = -i = e^{i3\pi/2}$, $\sqrt{z} = e^{i3\pi/4} = \frac{-1+i}{\sqrt{2}}$

$$\mathrm{Res} = \lim_{z\to e^{3\pi i/2}} \frac{\sqrt{z}}{z-i} = \frac{e^{i3\pi/4}}{-2i} = \frac{-1+i}{-2\sqrt{2}\,i} = \frac{-1-i}{2\sqrt{2}}$$

$$\Rightarrow 2I = 2\pi i\left(\frac{1-i}{2\sqrt{2}} - \frac{1+i}{2\sqrt{2}}\right) = 2\pi i\left(\frac{-2i}{2\sqrt{2}}\right)$$

$$\Rightarrow I = \pi/\sqrt{2}$$

(6)



$$\oint \frac{e^{-i\omega z}}{\cosh z}\,dz = 2\pi i \cdot \operatorname{Res}\left(\frac{\pi i}{2}\right) = 2\pi e^{\pi\omega/2}$$

$$\cosh z = 0 \Rightarrow e^{z}+e^{-z}=0 \Rightarrow e^{2z}=-1 \Rightarrow z=(n+\tfrac{1}{2})\pi i \ : \ \frac{\pi i}{2} \text{ in}$$

$$\operatorname{Res} = \lim_{z\to\pi i/2} (z-\pi i/2)\frac{e^{-i\omega z}}{\cosh z} = \left.\frac{e^{-i\omega z}}{\sinh z}\right|_{z=\pi i/2} = \frac{e^{\pi\omega/2}}{i}$$

$$\sinh\frac{\pi i}{2} = \frac{1}{2}\left(e^{\pi i/2}-e^{-\pi i/2}\right) = i$$

So: $$\int_{-R}^{R} + \int_{R+\pi i}^{-R+\pi i} + \int_{\uparrow} + \int_{\downarrow} = 2\pi e^{\pi\omega/2}$$

Now $$\int_{R+\pi i}^{-R+\pi i} = \int_{R}^{-R} \frac{e^{-i\omega(x+i\pi)}dx}{\cosh(x+i\pi)} = e^{\omega\pi}\int_{-R}^{R}\frac{e^{-i\omega x}}{\cosh x}\,dx$$

since $\cosh(x+i\pi) = -\cosh x$

Also $$\left|\int_{\uparrow}\right| \le \left|\int_{y=0}^{\pi} 2\frac{e^{-i\omega(R+iy)}\,i\,dy}{e^{R}e^{iy}+e^{-R}e^{-iy}}\right| \le \int_{0}^{\pi}\frac{2e^{\omega y}\,dy}{e^{R}-e^{-R}} \xrightarrow{R\to\infty} 0$$

and similarly for $\int_{\downarrow}$. So

$$(1+e^{\omega\pi})\, I = 2\pi e^{\pi\omega/2} \Rightarrow I = \frac{2\pi e^{\pi\omega/2}}{1+e^{\omega\pi}}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{-i\omega x}}{\cosh x}\,dx = \frac{\pi}{\cosh(\omega\pi/2)}$$