Set 6 Solutions

(1a) $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx, \quad \Gamma(\beta) = \int_0^\infty y^{\beta-1} e^{-y}\,dy$

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} e^{-(x+y)}\,dx\,dy$$

~~Let~~ $\cancel{x = r\cos^2\vartheta, \; y = r\sin^2\vartheta}$

$\cancel{r = x+y, \; \tan^2\vartheta = y/x \Rightarrow \tan\vartheta = (y/x)^{1/2}}$

Let $x = u^2,\; y = v^2,\; dx = 2u\,du,\; dy = 2v\,dv$

$$\Gamma(\alpha)\Gamma(\beta) = 4\int_0^\infty \int_0^\infty u^{2\alpha-2} v^{2\beta-2} e^{-(u^2+v^2)}\, u\, v\, du\, dv$$

$$= 4\int_0^\infty \int_0^\infty u^{2\alpha-1} u^{2\beta-1} e^{-(u^2+v^2)}\, du\, dv$$

Let $u = r\cos\vartheta,\; v = r\sin\vartheta$: $\quad du\,dv = r\,dr\,d\vartheta, \quad 0 \le \vartheta \le \frac{\pi}{2}, \quad 0 \le r < \infty$

$$\text{Int} = 4\int_0^\infty \int_0^{\pi/2} r^{2(\alpha+\beta)-2} e^{-r^2} \cos^{2\alpha-1}\vartheta \sin^{2\beta-1}\vartheta \; r\,dr\,d\vartheta$$

$$= 4\Bigg(\underbrace{\int_0^\infty r^{2(\alpha+\beta)-2} e^{-r^2} r\,dr}_{I_1}\Bigg)\Bigg(\underbrace{\int_0^{\pi/2} \cos^{2\alpha-1}\vartheta \sin^{2\beta-1}\vartheta\, d\vartheta}_{I_2}\Bigg)$$

But $\quad I_1 = \frac{1}{2}\int_0^\infty z^{(\alpha+\beta)-1} e^{-z}\,dz = \frac{1}{2}\Gamma(\alpha+\beta)$

So $\quad \Gamma(\alpha)\Gamma(\beta) = 2\Gamma(\alpha+\beta)\int_0^{\pi/2} \cos^{2\alpha-1}\vartheta \sin^{2\beta-1}\vartheta\, d\vartheta$

Finally, let $\cos^2\vartheta = t$, $-2\cos\vartheta\sin\vartheta\, d\vartheta = dt$

$1-t = \sin^2\vartheta$, $-2\cos\vartheta\sin\vartheta\, d\vartheta = -2\sqrt{t(1-t)}\, d\vartheta$

So
$$I_2 = -\frac{1}{2}\int_1^0 t^{\alpha-1/2}(1-t)^{\beta-1/2}\cdot t^{-1/2}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{2}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

Finally
$$\boxed{\Gamma(\alpha)\,\Gamma(\beta) = \Gamma(\alpha+\beta)\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt}$$

(1b)
$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \quad (\beta = 1-\alpha)$$

$$= \int_0^1 t^{\alpha-1}(1-t)^{-\alpha}dt = \int_0^1\left(\frac{t}{1-t}\right)^{\alpha}\frac{dt}{t}$$

let $t/_{1-t} = e^u$, $e^u du = \left(\frac{1}{1-t} + \frac{t}{(1-t)^2}\right)dt$

$$\Rightarrow e^u du = \frac{dt}{(1-t)^2} \Rightarrow \frac{t}{1-t}du = \frac{dt}{(1-t)^2} \Rightarrow (1-t)\,du = \frac{dt}{t}$$

Or $\frac{t}{1-t} = e^u \Rightarrow t = (1-t)e^u \Rightarrow t + te^u = e^u$

$$\Rightarrow t = \frac{e^u}{1+e^u}, \quad 1-t = \frac{1}{1+e^u}$$

$$\Rightarrow \frac{dt}{t} = \frac{du}{1+e^u} : \quad \int_0^1\left(\frac{t}{1-t}\right)^{\alpha}\frac{dt}{t} = \int_{-\infty}^{\infty}\frac{e^{\alpha u}}{1+e^u}du$$

$t=0 \Rightarrow u=-\infty$, $t=1 \Rightarrow u=+\infty$

So $\Gamma(\alpha)\,\Gamma(1-\alpha) = \underbrace{\Gamma(\alpha+1-\alpha)}_{\substack{\Gamma(1)\\ \shortparallel \\ 1}} \int_{-\infty} \frac{e^{\alpha u}}{1+e^{u}}\,du$

From Homework problem 4(3c), this integral equals $\pi/\sin\pi\alpha$, so $\Gamma(\alpha)\,\Gamma(1-\alpha) = \dfrac{\pi}{\sin\pi\alpha}$

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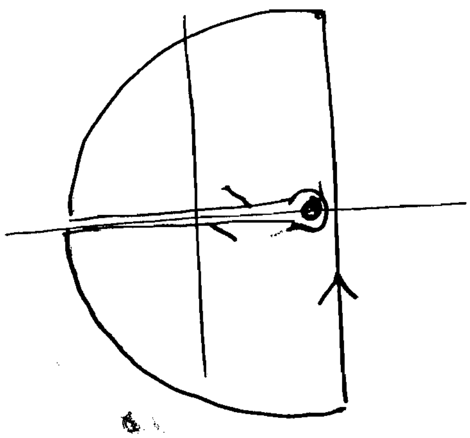
(2)



$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{(s-a)^{\nu+1}}\,ds \qquad \text{let } z = s-a,\ s = z+a$$

$$= e^{at}\cdot\underbrace{\frac{1}{2\pi i}\int_{(\gamma-a)-i\infty}^{(\gamma-a)+i\infty} \frac{e^{zt}}{z^{\nu+1}}\,dz}_{\shortparallel\ \frac{t^{\nu}}{\Gamma(\nu+1)}} \quad \text{done in class}$$

So $\boxed{f(t) = \dfrac{t^{\nu}e^{at}}{\Gamma(\nu+1)}}$

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(3) $sU - \underset{2}{\overset{}{u(0)}} + \frac{1}{s}U = \frac{1}{s}$

$\Rightarrow \left(s+\frac{1}{s}\right)U = 2+\frac{1}{s} \Rightarrow U(s) = \dfrac{2s}{s^2+1} + \dfrac{1}{s^2+1}$

$\Rightarrow u(t) = 2\cos t + \sin t$

(4.) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \underbrace{\int_0 f(t)e^{-st}\,dt}_{G(s)} +$

$$G(s) = \int_0^T f(t)\,e^{-st}\,dt\Big) \qquad + \int_T^\infty f(t)\,e^{-st}\,dt$$

But $\int_T^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t+T)\,e^{-s(t+T)}\,dt$

let $t \to t+T$

$$= e^{-sT}\int_0^\infty f(t)\,e^{-st}\,dt = e^{-sT}F(T)$$

So $F(T) = G(s) + e^{-sT}F(T) \Rightarrow \boxed{F(T) = \dfrac{G(s)}{1-e^{-sT}}}$

Ex: $f(t) = |\sin\omega t|$; let $g(t) = \begin{cases} \sin\omega t, & 0 \le t \le \frac{\pi}{\omega} \\ 0, & t > \frac{\pi}{\omega} \end{cases}$

Then $G(s) = \int_0^{\pi/\omega} \sin\omega t\, e^{-st}\,dt =$ (Dwight, "Tables of Integrals etc.")

$$= -\frac{e^{-st}}{s^2+\omega^2}\left(s\sin\omega t + \omega\cos\omega t\right)\Big|_0^{\pi/\omega}$$

$$= \cancel{\frac{\omega - s\,e^{-s}}{s^2+\omega^2}} \quad \cancel{\frac{\omega + s\,e^{s\pi/\omega}}{s^2+\omega^2}} = \frac{\omega}{s^2+\omega^2}\left(1+e^{-\frac{s\pi}{\omega}}\right)$$

So $F(t) = \dfrac{\omega}{s^2+\omega^2}\,\dfrac{1+e^{-s\pi/\omega}}{1-e^{-s\pi/\omega}} = \dfrac{\omega}{s^2+\omega^2}\coth\left(\dfrac{s\pi}{2\omega}\right)$

(5) $F(s) = \Phi(s) \cdot \cancel{\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}} \frac{\Gamma(1-\alpha)}{s^{1-\alpha}}$

$$\Rightarrow \Phi(s) = \frac{1}{\Gamma(1-\alpha)} \cdot s^{1-\alpha} F(s)$$

$$\Rightarrow \Phi(s) = s \frac{1}{\Gamma(1-\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} \frac{F(s)}{s^{\alpha}} = \cancel{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-1)} \frac{d}{dt} \int_0^t f(t)\, t^{\alpha}} dt$$

$$\phi(t) = \frac{\Gamma(\alpha+1)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(\tau)\,(t-\tau)^{\alpha}\, dt$$

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(6) $J_0(at) = \frac{1}{\pi} \int_0^{\pi} \cos(at\sin\theta)\, d\theta$

$\quad\; \shortparallel$

$\;\; f(t)$

$$F(s) = \mathcal{L}\{J_0(at)\} = \frac{1}{\pi} \int_0^{\infty} \left(\int_0^{\pi} \cos(at\sin\theta)\, d\theta \right) e^{-st}\, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} d\theta \underbrace{\int_0^{\infty} \cos(a\sin\theta \cdot t)\, e^{-st}\, dt}_{= \frac{s}{s^2 + a^2\sin^2\theta}}$$

$$= \frac{s}{\pi} \int_0^{\pi} \frac{d\theta}{s^2 + a^2 \underbrace{\sin^2\theta}_{= \frac{1}{2} - \frac{1}{2}\cos 2\theta}} = \frac{2s}{\pi} \int_0^{\pi} \frac{d\theta}{(2s^2 + a^2) - a^2\cos 2\theta}$$

$$= \frac{s}{\pi} \int_0^{2\pi} \frac{d\phi}{A - B\cos\phi}$$

$\underline{A = 2s^2 + a^2 > B = a^2}$

This was done in (p. 9.2) of notes: we saw that

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \quad |a|<1$$

Here $$\int_0^{2\pi} \frac{d\phi}{A - B\cos\phi} = \frac{1}{A}\int_0^{2\pi} \frac{d\phi}{1-\left(\frac{B}{A}\right)\cos\phi}, \quad a = -\left(\frac{B}{A}\right)$$

$A = 2s^2+a^2$

$B = a^2$

$$= \frac{1}{A} \cdot \frac{2\pi}{\sqrt{1-(B/A)^2}} = \frac{2\pi}{\sqrt{A^2-B^2}}$$

$$= \frac{2\pi}{\sqrt{4s^4+\cancel{a^4}+4s^2a^2-\cancel{a^4}}} = \frac{\pi}{s\cancel{a}\sqrt{a^2+s^2}}$$

So $$F(s) = \frac{\cancel{s}}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{\cancel{sa}\sqrt{a^2+s^2}} = \frac{1}{\sqrt{s^2+a^2}}$$

Clearly $$F(s)^2 = \frac{1}{s^2+a^2}$$

$$\Rightarrow \boxed{\frac{1}{a}\sin at = \int_0^t J_0(at)\, J_0(a(t-\tau))\, d\tau}$$

by convolution theorem.

(7) Laplace transform in time:

$$U(x,s) = \int_0^\infty u(x,t)\, e^{-st}\, dt$$

initially zero displacement, $= 0$

$$\Rightarrow \quad \frac{\partial^2}{\partial x^2}U - k^2 U = \frac{1}{c^2}\left(s^2 U - s\cancel{u(x,0)} - \cancel{u_t(x,0)}\right)$$

$= 0$ initially at rest

$$\Rightarrow \quad \boxed{U_{xx} - \left(k^2 + \frac{s^2}{c^2}\right)U = 0}$$

$$\Rightarrow \quad U(x,s) = A e^{-\sqrt{k^2+\frac{1}{c^2}s^2}\,x} + B e^{+\sqrt{k^2+\frac{1}{c^2}s^2}\,x}$$

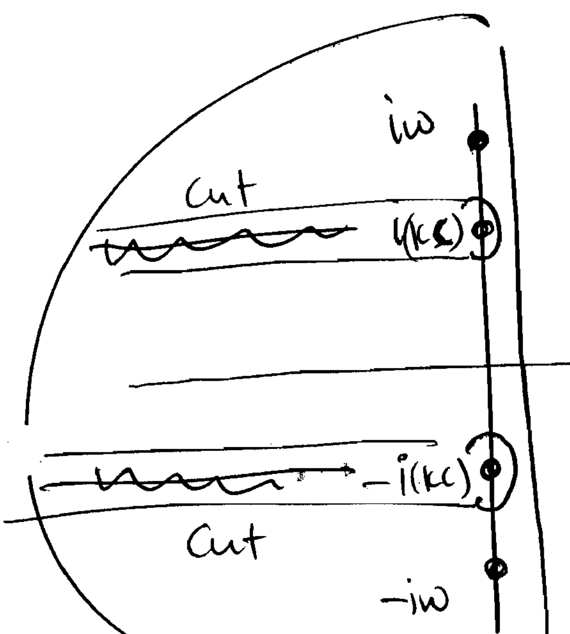
At $x=0$, $\quad U(0,s) = \dfrac{\omega}{s^2+\omega^2}$

while, as $x \to \infty$, $U(x,t) \to 0$

$$\Rightarrow \quad U(x,s) = \frac{\omega}{s^2+\omega^2}\, e^{-\frac{1}{c}\sqrt{s^2+(kc)^2}\,x}$$

↳ branch points: $s = \pm ikc$

↳ poles: $s = \pm i\omega$



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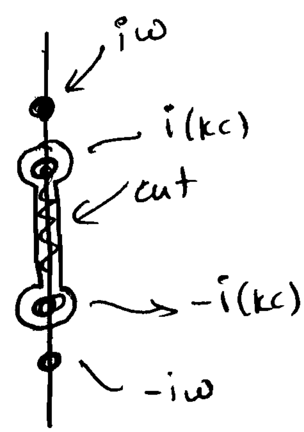

Find transform in terms of residues at $\pm i\omega$ and integral $\int_{-(kc)}^{(kc)}$ over imaginary axis.