Set 8 Solutions

⟨1⟩ (i) Let $w = z^{\alpha}\phi(z)$; substituting

$$w' = \alpha z^{\alpha-1}\phi + z^{\alpha}\phi' ; \quad w'' = \alpha(\alpha-1)z^{\alpha-2}\phi + 2\alpha z^{\alpha-1}\phi' + z^{\alpha}\phi''$$

$$\Rightarrow z^{\alpha+2}\phi'' + \left\{2\cancel{\alpha} + (1-\cancel{2\alpha})\right\} z^{\alpha+1}\phi' + \left\{\alpha(\cancel{\alpha-1})z^{\alpha} + \cancel{\alpha z^{\alpha}} + \beta^2\gamma^2 z^{2\gamma+\alpha} + (\cancel{\alpha^2} - \nu^2\gamma^2)z^{\alpha}\right\}\phi = 0$$

$$\Rightarrow z^2\phi'' + z\phi' + \gamma^2\left\{\beta^2 z^{2\gamma} - \nu^2\right\}\phi = 0.$$ Now change variables

to $u = \beta z^{\gamma} \Rightarrow \dfrac{d}{dz} = \dfrac{du}{dz}\dfrac{d}{du} = \beta\gamma z^{\gamma-1}\dfrac{d}{du}$; $\dfrac{d^2}{dz^2} = \dfrac{d}{dz}(\beta\gamma z^{\gamma-1})\dfrac{d}{du} + (\beta\gamma z^{\gamma-1})^2\dfrac{d^2}{du^2}$

$$\Rightarrow \frac{d^2}{dz^2} = \beta\gamma(\gamma-1)z^{\gamma-2}\frac{d}{du} + \beta^2\gamma^2 z^{2\gamma-2}\frac{d^2}{du^2}$$

$$\Rightarrow z^2\frac{d^2}{dz^2} + z\frac{d}{dz} = \beta\gamma(\gamma-1)z^{\gamma}\frac{d}{du} + \beta^2\gamma^2 z^{2\gamma}\frac{d^2}{du^2} + \beta\gamma z^{\gamma}\frac{d}{du} = \beta^2\gamma^2 z^{2\gamma}\frac{d^2}{du^2} + \beta\gamma^2 z^{\gamma}\frac{d}{du}$$

$$= \gamma^2\left\{u^2\frac{d^2}{du^2} + u\frac{d}{du}\right\}$$

So $$z^2\frac{d^2\phi}{dz^2} + z\frac{d}{dz} + \gamma^2\left\{\beta^2 z^{2\gamma} - \nu^2\right\}\phi = \gamma^2\left[u^2\frac{d^2\phi}{du^2} + u\frac{d\phi}{du} + (u^2-\nu^2)\phi\right] = 0$$

$$\Rightarrow \phi = A J_{\nu}(u) + B Y_{\nu}(u) \quad \Rightarrow w(z) = A z^{\alpha} J_{\nu}(\beta z^{\gamma}) + B z^{\alpha} Y_{\nu}(\beta z^{\gamma})$$

[see end of writeup for the rest of this problem].

⟨2⟩ Assume $\nu \neq n$, integer, so $J_{\nu}, J_{-\nu}$ independent and we can use the series expressions $(\nu > 0)$

$$J_{\nu} = \left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{\underset{\Gamma(k+1)}{\overset{\parallel\!\!\:|}{k!}}\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}$$

$$J_{-\nu} = \left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(-\nu+k+1)}\left(\frac{z}{2}\right)^{2k}$$

The wronskian $W(J_{\nu}, J_{-\nu}) = \begin{vmatrix} J_{\nu} & J_{-\nu} \\ J_{\nu}' & J_{-\nu}' \end{vmatrix} = J_{\nu}J_{-\nu}' - J_{-\nu}J_{\nu}'$

satisfies a differential equation (see notes p. 13.8)

$$\frac{dW}{dz} = -\frac{1}{z}W \Rightarrow W = \frac{A}{z},$$ A some constant.

To determine A, consider $\lim\limits_{z\to 0}(zW)$. Now

$$J_{\nu}' = \frac{1}{2}\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+\nu)}{\Gamma(k+1)\Gamma(-\nu+k+1)}\left(\frac{z}{2}\right)^{2k-1}$$

$$J_{-\nu}' = \frac{1}{2}\left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^{\infty}\frac{(-1)^k(2k-\nu)}{\Gamma(k+1)\Gamma(-\nu+k+1)}\left(\frac{z}{2}\right)^{2k-1}$$

So $\lim\limits_{z\to 0} z J_\nu J_{-\nu}' = \dfrac{-\nu}{\Gamma(1)\Gamma(\nu+1)\Gamma(1)\Gamma(-\nu+1)} = -\dfrac{1}{\Gamma(\nu)\Gamma(1-\nu)}$

$\lim\limits_{z\to 0} z J_{-\nu} J_\nu' = \dfrac{1}{\Gamma(\nu)\Gamma(1-\nu)}$ $\qquad\therefore\ \boxed{W = -\dfrac{2\sin\pi\nu}{\pi z}}$

$\Rightarrow\ \lim\limits_{z\to 0} z W = -\dfrac{2}{\Gamma(\nu)\Gamma(1-\nu)} = -\dfrac{2\sin\pi\nu}{\pi} = A$

(When ν = integer, this gives $W(J_\nu, J_{-\nu}) = 0$, as expected).

For $W(J_\nu, Y_\nu) = W\left(J_\nu, \frac{1}{\sin\pi\nu}\{J_\nu\cos\pi\nu - J_{-\nu}\}\right) =$

$$= \frac{\cos\pi\nu}{\sin\pi\nu}\underbrace{W(J_\nu, J_\nu)}_{=0} - \frac{1}{\sin\pi\nu} W(J_\nu, J_{-\nu}) = \frac{2}{\pi z}$$

<3> (i) $\dfrac{d}{dz}\left(z^n J_n\right) = n z^{n-1} J_n + z^n J_n'$; (we have $J_n' = \frac{1}{2}(J_{n-1} - J_{n+1})$)

$= z^n\left\{\frac{n}{z} J_n + J_n'\right\}$ $\qquad \frac{2n}{z} J_n = J_{n-1} + J_{n+1}$

$= z^n\left\{\frac{1}{2}J_{n-1} + \cancel{\frac{1}{2}J_{n+1}} + \frac{1}{2}J_{n-1} - \cancel{\frac{1}{2}J_{n+1}}\right\} = z^n J_{n-1}$

(ii) $\dfrac{d}{dz}\left(z^{-n} J_n\right) = -n z^{-n-1} J_n + z^{-n} J_n'$

$= z^{-n}\left\{-\frac{n}{z} J_n + J_n'\right\} = z^{-n}\left\{-\cancel{\frac{1}{2}J_{n-1}} - \frac{1}{2}J_{n+1} + \cancel{\frac{1}{2}J_{n-1}} - \frac{1}{2}J_{n+1}\right\} = -z^{-n} J_{n+1}$

Rolle's thm. states that if $f(x)$ is continuously differentiable in some interval $[a, b)$ and $f(x_1) = f(x_2) = 0$, $a < x_1 < x_2 < b$ then $\exists\ x_0$: $f'(x_0) = 0$ with $x_1 < x_0 < x_2$. In the second identity, let $z = x$ real, $x > 0$. and let $f(x) \equiv x^{-n} J_n(x)$
Let $J_n(x_1) = J_n(x_2) = 0$, $0 < x_1 < x_2$. Then $\exists\ x_0$: $x_1 < x_0 < x_2$ with $\left.\dfrac{d}{dx}\left(x^{-n} J_n(x)\right)\right|_{x=x_0} = -x_0^{-n} J_{n+1}(x_0) = 0 \Rightarrow J_{n+1}(x_0) = 0$.

⟨4⟩ Note : using the generating function we proved the recurrence relations (i) $J_{n+1} = \frac{n}{z} J_n - J_n'$, (ii) $\frac{2n}{z} J_n = J_{n-1} + J_{n+1}$, (iii) $2J_n' = J_{n-1} - J_{n+1}$ for integer n. Since $J_n(z)$ is an entire function of n (this can be seen from the integral representation), these relations hold for all n. Thus, using $Y_\nu = \frac{1}{\sin\nu\pi}\left[\cos\nu\pi\, J_\nu - J_{-\nu}\right]$; $\begin{pmatrix} \sin(\nu+1)\pi = -\sin\nu\pi \\ \cos(\nu+1)\pi = -\cos\nu\pi \end{pmatrix}$

$$\left.\begin{aligned} Y_{\nu+1} &= \frac{1}{\sin\nu\pi}\left[\cos\nu\pi\, J_{\nu+1} + J_{-\nu-1}\right] \\ \frac{\nu}{z} Y_\nu &= \cot\nu\pi \left(\frac{\nu}{z} J_\nu\right) - \frac{1}{\sin\nu\pi}\left(\frac{\nu}{z}\right) J_{-\nu} \\ Y_\nu' &= \cot\nu\pi \left(J_\nu' \right) - \frac{1}{\sin\nu\pi}\left(J_{-\nu}'\right) \end{aligned}\right\}$$

$$\Rightarrow Y_{\nu+1} - \frac{\nu}{z} Y_\nu + Y_\nu' = \cot\nu\pi \left\{ J_{\nu+1} - \frac{\nu}{z} J_\nu + J_\nu' \right\} + \frac{1}{\sin\nu\pi}\left\{ J_{-\nu-1} + \frac{\nu}{z} J_{-\nu} - J_{-\nu}' \right\}$$

The first bracket vanishes by (i). For the second, consider

$$J_n' = \frac{1}{2}\left(J_{n-1} - J_{n+1}\right) , \quad \frac{2n}{z} J_n = J_{n-1} + J_{n+1} \Rightarrow J_{n+1} = \frac{2n}{z} J_n - J_{n-1}$$

$$\cancel{\Rightarrow J_n' = \frac{1}{2}\left(\frac{2n}{z} J_n - J_{n+1} - J_{n+1}\right) = \frac{n}{z} J_n - J_{n+1} \cdots}$$

$$J_n' = \frac{1}{2}\left(J_{n-1} - \frac{2n}{z} J_n + J_{n-1}\right) = J_{n-1} - \frac{n}{z} J_n \Rightarrow \underline{J_{n-1} - \frac{n}{z} J_n - J_n' = 0}$$

Settin $n = -\nu$ in this we get $J_{-\nu-1} + \frac{\nu}{z} J_\nu - J_{-\nu}' = 0$, so the second bracket also vanishes. Similarly

$$\begin{aligned} 2 Y_\nu' &= \cot\nu\pi \left(2 J_\nu'\right) - \frac{1}{\sin\nu\pi}\left(2 J_{-\nu}'\right) \\ Y_{\nu-1} &= \frac{1}{\sin\nu\pi}\left[\cos\nu\pi\, J_{\nu-1} + J_{-\nu+1}\right] \\ Y_{\nu+1} &= \frac{1}{\sin\nu\pi}\left[\cos\nu\pi\, J_{\nu+1} + J_{-\nu-1}\right] \end{aligned}$$

$$\Rightarrow 2Y_\nu' - Y_{\nu-1} + Y_{\nu+1} = \cot\nu\pi \left\{ 2J_\nu' - J_{\nu-1} + J_{\nu+1} \right\} - \frac{1}{\sin\nu\pi}\left\{ 2J_{-\nu}' - J_{-\nu+1} + J_{-\nu-1} \right\}$$

and both brackets vanish by virtue of $\underline{2J_n' = J_{n-1} - J_{n+1}}$

⟨5⟩ The radial part of Laplace eqn. in 3 d is

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$ Then, for $u = u(r)$ we have

$$\nabla^2 u + B^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + B^2 u = 0$$

Comparing with problem (1) we have $\alpha = -1/2$, $\beta = B$, $\gamma = 1$, $\nu = \pm\frac{1}{2}$

so that $u = A_1 z^{-1/2} J_{1/2}(Bz) + A_2 z^{-1/2} J_{-1/2}(Bz)$. From the series expansions we have

$$J_{1/2}(z) = \left(\frac{z}{2}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+1/2)} \left(\frac{z}{2}\right)^{2k}$$

Since $\Gamma(k+1+1/2) = (k+1/2)\Gamma(k+1/2) = \cdots = (k+\frac{1}{2})(k-\frac{1}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2})$

$$\left.\begin{array}{l}\Rightarrow 2^{k+1}\Gamma(k+1+\frac{1}{2}) = (2k+1)(2k-1)\cdots 1 \cdot \sqrt{\pi} \\ \text{while } 2^k\Gamma(k+1) = 2^k k! = (2k)(2k-2)\cdots 2\end{array}\right\}$$

$$2^{2k+1}\Gamma(k+1)\Gamma(k+1+\tfrac{1}{2}) = (2k+1)!\sqrt{\pi}$$

we have $J_{1/2}(z) = \left(\frac{2z}{\pi}\right)^{1/2} \frac{1}{z} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = \sqrt{\frac{2z}{\pi}}\ \frac{\sin z}{z}$

and similarly, $J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\cos z$, giving for u:

$$u(r) = A_1 \cdot \sqrt{\frac{2B}{\pi}}\ \frac{\sin Br}{Br} + A_2 \sqrt{\frac{2}{\pi B}}\ \frac{\cos Br}{r}$$

Since the solution is regular at the origin, $A_2 = 0$.

For $u(R) = 0 \Rightarrow \sin BR = 0 \Rightarrow BR = n\pi$; smallest value is $B = \pi/R$

$$u = A_1 \sqrt{\frac{2}{R}} \cdot \frac{R}{\pi r} \sin\left(\frac{\pi r}{R}\right)$$

$\langle 6 \rangle$ Using the recurrence relations

$$\text{(i)} \quad (2n+1)\, x P_n = (n+1) P_{n+1} + n P_{n-1} \quad \Rightarrow \left(\tfrac{d}{dx}\text{(i)}\right)$$

$$\text{(ii)} \quad (2n+1) P_n + (2n+1) x P_n' = (n+1) P_{n+1}' + n P_{n-1}'$$

$$\text{(iii)} \quad P_n + 2x P_n' = P_{n+1}' + P_{n-1}'$$

$\Rightarrow 2$ ~~[illegible]~~ gives

$2 * \text{(ii)} - (2n+1) * \text{(iii)}$ gives

$$\text{(iv)} \quad (2n+1) P_n = P_{n+1}' - P_{n-1}' \qquad (\text{part } b')$$

— . —

(a) in (i), set $x = 0$; use $P_0 = 1$, $P_1 = x$. now

$$(n+1) P_{n+1}(0) = -n P_{n-1}(0) \; ; \quad \text{since } P_0(0) = 1,\; P_1(0) = 0$$

$$\Rightarrow P_{2n+1}(0) = 0 \quad \text{while}$$

$$P_{2n}(0) = -\frac{2n-1}{2n} P_{2(n-1)}(0) = \cdots = (-1)^n \frac{(2n-1)\cdots 3 \cdot 1}{(2n)[2(n-1)]\cdots 2} P_0(0)$$

$$= (-1)^n \frac{(2n)!}{[(2n)(2n-2)\cdots 2]^2} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$$

$$\text{(C)} \int_0^1 P_n \, dx = \frac{1}{2n+1} \int_0^1 P_{n+1}' \, dx - \frac{1}{2n+1} \int_0^1 P_{n-1}' \, dx$$

Using part (b)

$$= \frac{1}{2n+1}\left[\cancel{P_{n+1}(1)} - P_{n+1}(0)\right] - \frac{1}{2n+1}\left[\cancel{P_{n-1}(1)} - P_{n-1}(0)\right]$$

$$\Rightarrow \int_0^1 P_n(x)\, dx = \frac{1}{2n+1}\left(P_{n-1}(0) - P_{n+1}(0)\right) = \begin{cases} 0, & n = 2m,\ n > 0 \\ \dfrac{1}{4m+3}\left[\dfrac{(-1)^m (2m)!}{2^{2m}(m!)^2} - \dfrac{(-1)^{m+1}(2m+2)!}{2^{2m+2}[(m+1)!]^2}\right] & n = 2m+1 \end{cases}$$

$n \neq 0$; $\; n-1 \to 2m$, $\; n+1 \to 2m+2$

$$\int_0^1 P_0(x)\, dx = \int_0^1 dx = 1$$

$$= \frac{(-1)^m}{4m+3}\left[\frac{(2m)!\, 2^2 (m+1)^{\cancel{2}} + (2m)!\,(2m+1)\, 2\, \cancel{(m+1)}}{2^{2m+2}\,(m+1)!\,(m)!\,\cancel{(m+1)}}\right]$$

$$= (-1)^m \left[\frac{(2m)!}{2^{2m+1}(m+1)!\, m!}\right] ; \quad n = 2m+1$$

⟨1⟩ continued

(i) $w'' - zw = 0 \Rightarrow z^2 w'' - z^3 w = 0$

$(1-2\alpha) = 0 \Rightarrow \alpha = 1/2$

$\beta^2\gamma^2 z^{2\gamma} + 1/4 - \nu^2\gamma^2 = \cancel{-}z^3 \Rightarrow \begin{cases} 2\gamma = 3 \Rightarrow \gamma = 3/2 \\ \beta = 2/3\, i \; ; \quad \nu = \frac{1}{2}\cdot\frac{2}{3} = 1/3 \end{cases}$

$$\therefore \quad w = A z^{1/2} J_{1/3}\left(\frac{2i}{3} z^{3/2}\right) + B z^{1/2} J_{1/3}\left(\frac{2i}{3} z^{3/2}\right)$$

(ii) $zw'' - w' + 4z^3 w = 0 \Rightarrow z^2 w'' - zw' + 4z^4 w = 0$

$1 - 2\alpha = -1 \Rightarrow \alpha = 1$

$\beta^2\gamma^2 z^{2\gamma} + 1 - \nu^2\gamma^2 = 4z^4 \Rightarrow \begin{cases} \gamma = 2 \\ \beta = 1 \end{cases} \quad \nu = \frac{1}{2}$

$$w = A z J_{1/2}(z^2) + B z Y_{1/2}(z^2)$$

Since $J_{1/2}(z) = \sqrt{\frac{2z}{\pi}}\, \frac{\sin z}{z} \Rightarrow J_{1/2}(z^2) = \sqrt{\frac{2}{\pi}}\, z\, \frac{\sin z^2}{z^2}$

$$Y_{1/2}(z) = \frac{1}{\sin \pi/2}\left(J_{1/2}(z) \cos \pi/2 - J_{-1/2}(z)\right) = -J_{-1/2}(z)$$

$$= -\sqrt{\frac{2}{\pi z}} \cos z$$

$$\Rightarrow Y_{1/2}(z^2) = -\sqrt{\frac{2}{\pi}}\, \frac{\cos z^2}{z}$$

So: $w(z) = A \sin z^2 + B \cos z^2$ (absorbing constant factors into A, B)

Problem Set 6 (Correction)

#5 Abel's integral equation for the function $\phi(t)$, $t>0$ is $\int_0^t \phi(\tau)(t-\tau)^{-\alpha}d\tau = f(t)$, $t>0$ where α is a constant, $0<\alpha<1$, and $f(t)$ is given. Solve for $\phi(t)$ using Laplace transforms.

Since $\mathcal{L}\{t^\beta\} = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$, $\beta>-1 \Rightarrow \beta+1>0$ (this restriction set so that $\int_0^\infty t^\beta e^{-st}dt$ exists, but also to avoid the poles of $\Gamma(z)$ at $z=-n$), we have, by taking Laplase transforms of both sides and using the convolution theorem:

$$F(s) = \Phi(s)\,\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}$$

$$\Rightarrow \quad \Phi(s) = \frac{1}{\Gamma(1-\alpha)}\, sF(s)\cdot\frac{1}{s^\alpha}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\cdot\left\{[sF(s)-f(0)]+f(0)\right\}\frac{\Gamma(\alpha)}{s^\alpha}$$

$$\Rightarrow \phi(t) = \frac{\sin\pi\alpha}{\pi}\cdot\left\{\int_0^t f'(\tau)(t-\tau)^{\alpha-1}d\tau + f(0)\,t^{\alpha-1}\right\}$$

However this assumes that $f'(t)$ exists, which was not given. Instead we write

$$\Phi(s) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\, s \cdot \left\{ F(s) \cdot \frac{\Gamma(\alpha)}{s^{\alpha}} \right\}$$

Now $$F(s) \cdot \frac{\Gamma(\alpha)}{s(\alpha)} = \mathcal{L}\left\{ \int_0^t f(\tau)\,(t-\tau)^{1-\alpha}\, d\tau \right\}$$

$$= \mathcal{L}\{ g(t) \}$$

with $g(0) = 0$, Thus

$$\Phi(s) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \left(s \cdot G(s) - g(0) \right)$$

$$\Rightarrow \phi(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \frac{d}{dt}\, g(t)$$

$$\Rightarrow \boxed{\phi(t) = \frac{\sin \pi\alpha}{\pi}\, \frac{d}{dt} \int_0^t f(\tau)\,(t-\tau)^{\alpha-1}\, d\tau}$$

which exists, provided $f(\tau)$ is continuous only.