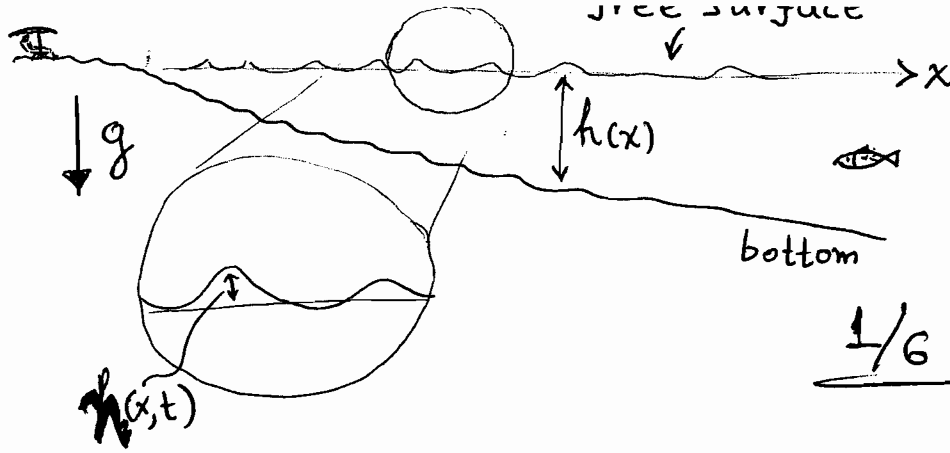
Set 4 Solutions (3)



$$\frac{\partial}{\partial x}\left(h(x)\frac{\partial \eta}{\partial x}\right) = \frac{1}{g}\frac{\partial^2 \eta}{\partial t^2}$$

(gravity waves - no surface tension effects (long wavelength)).

$h(x) = Kx$

Let $\eta(x,t) = e^{i\omega t} y(x)$; then

$$\partial_x\left(Kx\frac{\partial y}{\partial x}\right)\cancel{e^{i\omega t}} = -\frac{\omega^2}{g} y\cancel{e^{i\omega t}} \Rightarrow$$

$$Kx\,y'' + K y' + \frac{\omega^2}{g} y = 0$$

$$\Rightarrow \left(y'' + \frac{1}{x} y' + \frac{\omega^2}{Kgx} y = 0\right) x^2$$

Recall (Set 8, #1) that the equation

$$z^2 w'' + (1-2\alpha) z w' + (\beta^2\gamma^2 z^{2\gamma} + \alpha^2 - \nu^2\gamma^2) w = 0$$

has solution

$$w(z) = A z^\alpha J_\nu(\beta z^\gamma) + B z^\alpha Y_\nu(\beta z^\gamma)$$

(with A, B arbitrary)

Here; $1-2\alpha = 1$, $2\gamma = 1$, $\beta^2\gamma^2 = \frac{\omega^2}{gk}$, $\alpha^2 - \nu^2\gamma^2 = 0$

$\Rightarrow \alpha = 0$, $\gamma = 1/2$, $\beta = 2\omega/\sqrt{gk}$, $\nu = 0$

So: $y(x) = A\, J_0\left(\frac{2\omega}{\sqrt{kg}}\, x^{1/2}\right)$

and $\eta(x,t) \sim A\, J_0\left(\frac{2\omega}{\sqrt{kg}}\, x^{1/2}\right) \cos\omega t$

For x large:

recall $J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{\pi}{4}\right]$

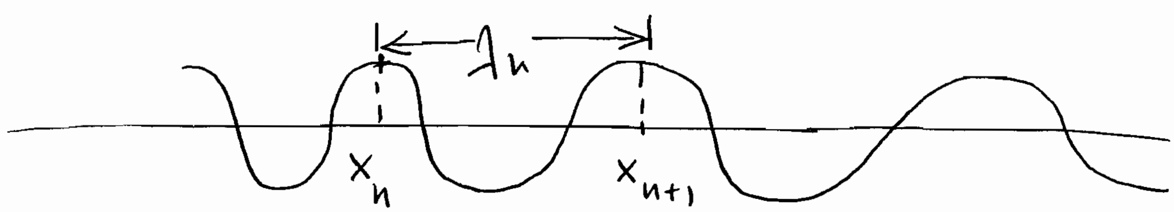
So $\eta(x,t) \sim A\left(\frac{1}{\pi\omega}\sqrt{kg}\right)^{1/2} x^{-1/4} \cos\left[\frac{2\omega}{\sqrt{kg}}\, x^{1/2} - \frac{\pi}{4}\right] \cos\omega t$

The wavelength is defined as the length over which we obtain a full period of the wave;

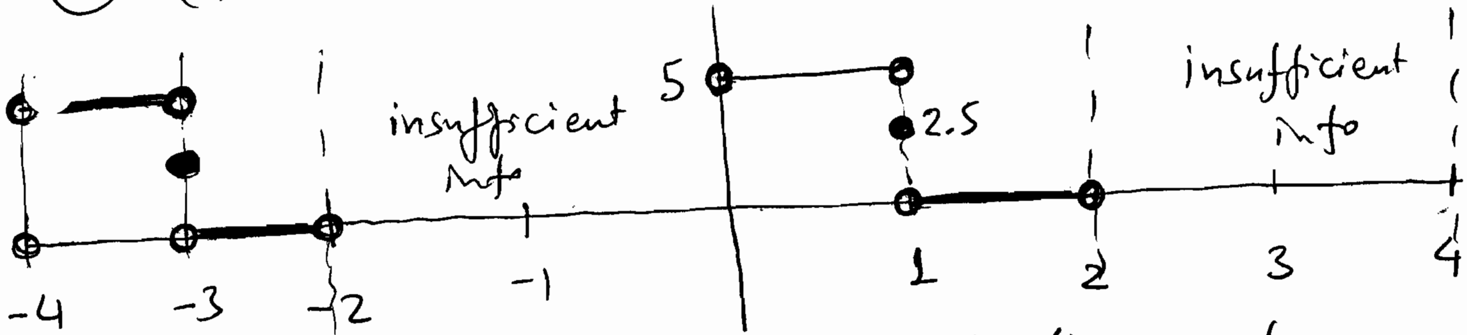
we set $\frac{\cancel{2}\omega}{\sqrt{kg}}\, x_n^{1/2} = \cancel{2} n\pi$

$\Rightarrow x_n = (n\pi)^2\, kg/\omega$

$\Rightarrow \Delta x_n = x_{n+1} - x_n = \boxed{(2n+1)\frac{\pi^2 kg}{\omega} = \lambda_n}$

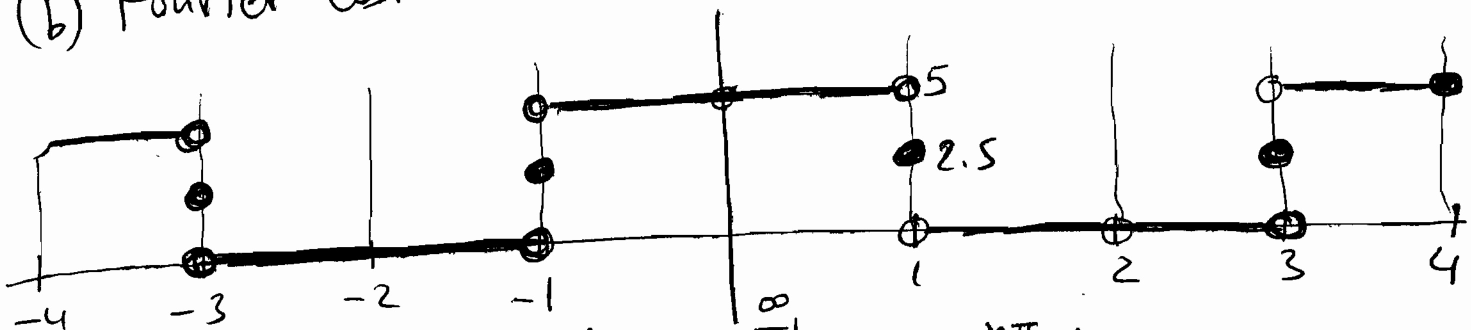


1 (a) Fourier



Since we want a function of period 4, and $f(x)$ is only given in $0<x<2$, we cannot assume anything about f on $-2<x<0$ and so cannot give the Fourier series coefficients.

(b) Fourier Cosine: we construct even periodic extension



$$f(x) = \cancel{\sum_{n=1}} \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos \frac{n\pi}{2} x$$

$$a_n = \int_0^2 f(x) \cos \frac{n\pi}{2} x \, dx = 5 \int_0^1 \cos \frac{n\pi}{2} x \, dx = \frac{10}{n\pi} \sin \frac{n\pi}{2} x \Big|_0^1$$

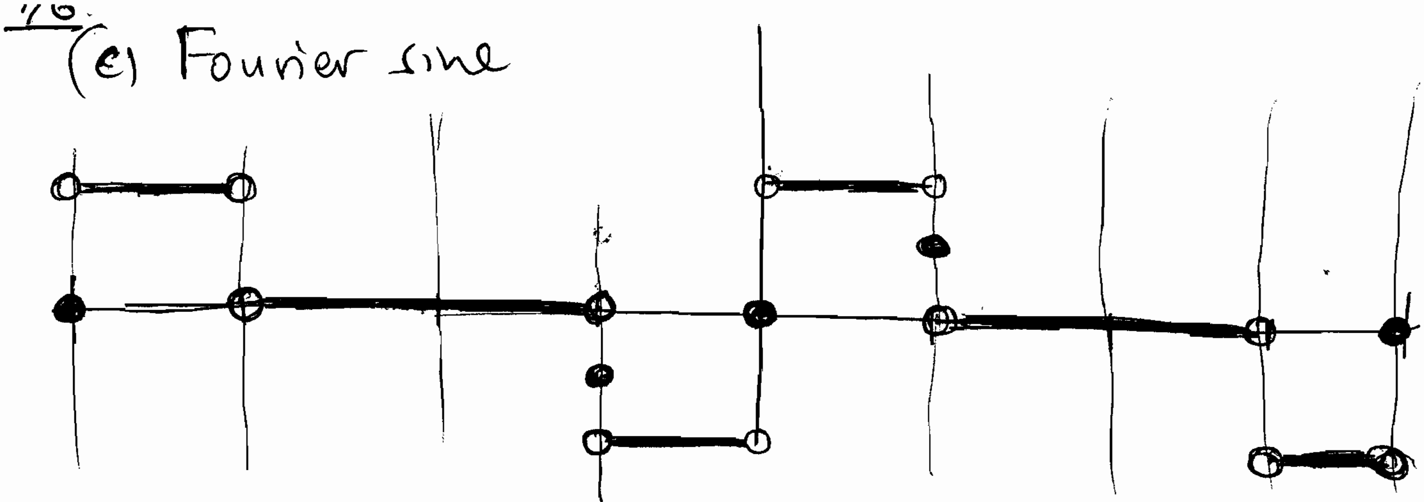
$$\Rightarrow \quad a_n = \frac{10}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 0, & n = 2m \\ (-1)^m \dfrac{10}{(2m+1)\pi}, & n = 2m+1 \end{cases}$$

$$a_0 = 5$$

$$f(x) = 2.5 + \frac{10}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{2m+1} \cos \frac{(2m+1)\pi}{2} x$$

10.

(e) Fourier sine



(odd periodic extension)

$$f(x) = \sum_{1}^{\infty} b_n \sin \frac{n\pi}{2} x$$

$$b_n = \int_0^2 f(x) \sin \frac{n\pi}{2} x \, dx = 5\int_0^1 \sin \frac{n\pi}{2} x \, dx$$

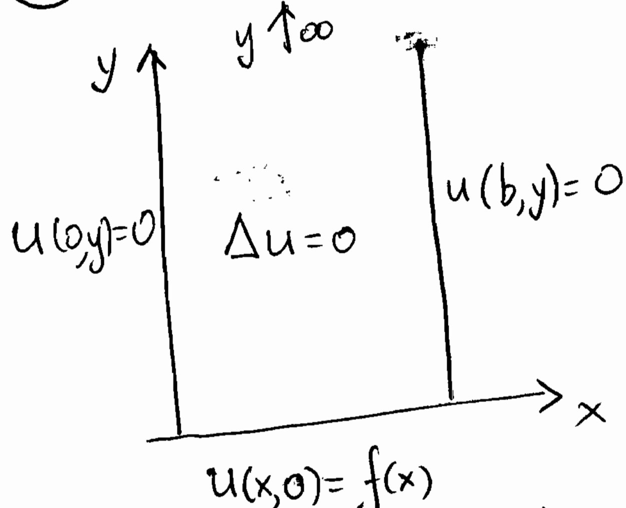
$$= -\frac{10}{n\pi} \cos \frac{n\pi}{2} x \Big|_0^1$$

$$= -\frac{10}{n\pi} \left(\cos \frac{n\pi}{2} - 1 \right)$$

$$\cos \frac{n\pi}{2} = \begin{cases} (-1)^m, & n = 2m \\ 0, & n = 2m+1 \end{cases}$$

i.e. $$b_n = \begin{cases} -\frac{10}{2m\pi} \left((-1)^m - 1 \right), & n = 2m \\ \frac{10}{(2m+1)\pi}, & n = 2m+1 \end{cases}$$

(2) $|u|<\infty$



Use a sine series in x; write

$$u(x,y) = \sum_{n=1}^{\infty} u_n(y) \sin\left(\frac{n\pi}{b} x\right)$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{b}\right) x$$

where $u_n(y) = \frac{2}{b}\int_0^b u(x,y) \sin \frac{n\pi}{b} x \, dx$, $f_n = \frac{2}{b}\int_0^{\infty} f(x) \sin \frac{n\pi}{b} x \, dx$

Multiply d.e. by $\sin \frac{n\pi}{b} x$ and integrate $\frac{2}{b}\int_0^b dx$ — :

$$\frac{2}{b}\int_0^b u_{xx}(x,y) \sin \frac{n\pi}{b} x \, dx + \frac{d^2}{dy^2}\underbrace{\left\{ \frac{2}{b}\int_0^b u(x,y) \sin \frac{n\pi}{b} x \, dx \right\}}_{= u_n(y)} = 0$$

$\downarrow$ integrate by parts:

$$\frac{2}{b} u_x \underbrace{\sin \frac{n\pi}{b} x \Big|_0^b}_{=0} - \frac{2}{b}\int_0^b u_x \left(\sin \frac{n\pi}{b} x\right)' dx = -\frac{2}{b}\left(\frac{n\pi}{b}\right)\int_0^b u_x \cos \frac{n\pi}{b} x \, dx$$

$$= -\frac{2}{b}\left(\frac{n\pi}{b}\right) u(x,y) \cos \frac{n\pi}{b} x \Big|_0^b + \frac{2}{b}\left(\frac{n\pi}{b}\right)\int_0^b u \left(\cos \frac{n\pi}{b} x\right)'$$

$$= -\frac{2}{b}\left(\frac{n\pi}{b}\right) \underbrace{u(b,y)}_{=0} (-1)^n + \left(\frac{2}{b}\right)\frac{n\pi}{b} \underbrace{u(0,y)}_{=0}$$

$$\circledast \quad = -\frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \int_0^b u(x,y) \sin \frac{n\pi}{b} x \, dx = -\left(\frac{n\pi}{b}\right)^2 u_n(y)$$

$\longrightarrow$

So $$\frac{d^2 u_n(y)}{dy^2} - \left(\frac{n\pi}{b}\right)^2 u_n(y) = 0$$

$$\Rightarrow u_n(y) = A_n e^{-\left(\frac{n\pi}{b}\right)y} + B_n e^{\left(\frac{n\pi}{b}\right)y}$$

Since $|u(x,y)| < \infty$ as $y \to \infty$ it follows that the $u_n(y)$ cannot grow $\Rightarrow B_n = 0$.

So $u_n(y) = A_n e^{-\left(\frac{n\pi}{b}\right)y}$. Now

$$u(x,y) = \sum_{n=1}^{\infty} u_n(y) \sin \frac{n\pi}{b} x \; ; \; u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi}{b} x$$

$$\Rightarrow u_n(0) = f_n \Rightarrow A_n = f_n$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} f_n e^{-\left(\frac{n\pi}{b}\right)y} \sin \frac{n\pi}{b} x$$

(*) If instead we had, say, $u(b,y) = g(y)$, then

$$\frac{2}{b}\int_0^b u_{xx} \sin \frac{n\pi}{b} x \, dx = -\frac{2}{b}\left(\frac{n\pi}{b}\right) u(b,y) (-1)^n - \left(\frac{n\pi}{b}\right)^2 u_n(y)$$

$$= -\frac{2}{b}\left(\frac{n\pi}{b}\right)(-1)^n g(y) - \left(\frac{n\pi}{b}\right)^2 u_n(y)$$

So : $$\begin{cases} \dfrac{d^2 u_n}{dy^2} - \left(\dfrac{n\pi}{2}\right)^2 u_n = \dfrac{2}{b}\left(\dfrac{n\pi}{b}\right)(-1)^n g(y) \\ u_n(0) = f_n \, , \quad |u_n(\infty)| < \infty \end{cases}$$

which can be solved by variation of parameters etc.