Math. 466

Methods of Applied Mathematics

Vagéli Coútsias

vageli @ math.unm.edu

office (505) 277-3310

(temp. office hours: T-Th. ~~3:15-4:45~~) 10:45-12:15

Fax: (505) 277-5505

Sec. " 277-4613

Office: Hum. 432

TEXT: Mathematical Methods
for Physicists, $6^{th.}$ ed.

G. Arfken & H. Weber

(topics from Ch. 5-16)

Grading — Homework (12 sets) 60%

Exams will be open book/notes (Take home - work alone!)

Midterms: I (Oct. 9) 10%

II (Nov. 20) 10%

Final (Dec. 11) 20%

Fundamental Problems

* Integral Equations (Ch. 16)

$$f(x) = \int_a^b K(x,t)\,\phi(t)\,dt$$

Fredholm, 1st kind

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\,\phi(t)\,dt$$

($b \to x$: Volterra) Fredholm, 2nd kind

- Differential equations can be recast as integral equations. (converse not always possible)

$y'' + \omega^2 y = 0$, $y(0)=0,\ y'(0)=1$ $\longrightarrow$ $y(x) = \omega^2 \int_0^x (t-x)\,y(t)\,dt + x$

$y(0) = y(b) = 0$ $\longrightarrow$ $y(x) = \omega^2 \int_0^b K(x,t)\,y(t)\,dt$; $K = ?$

*Sources of problems

Electromagnetism (Maxwell equs)

$$\nabla\cdot D = \rho \qquad \nabla\times E = -\frac{\partial B}{\partial t}$$

$$\nabla\cdot B = 0 \qquad \nabla\times H = \frac{\partial D}{\partial t} + J$$

$$(D = \varepsilon_0 E,\ B = \mu_0 H \text{ in vacuum})$$

Diffusion (heat conduction, mixing)

$$\frac{\partial c}{\partial t} = \nabla\cdot(D\nabla c) + f(c)$$

↳ chem. reaction

Quantum mechanics

$$E\psi = \qquad\qquad$$

$$i\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi \qquad \text{(Schrödinger equ.)}$$

Fluid flow

$$\frac{\partial u}{\partial t} + u\cdot\nabla u = \frac{1}{\rho}\nabla p + \nu\Delta u \qquad \text{(Navier-Stokes)}$$

Potential equ: $\Delta u = f$

(heat from stationary sources, potential for fixed charges, shape of membrane under constant force)

Helmholtz equ: $\Delta u + k^2 u = f$

(steady propagation of monochromatic waves)

Wave equation: $\Box u = g$

(unidirectional waves in vacuum) $\Box = \partial_t - |\nabla \cdot|$

Conservation laws (mass, charge, momentum, energy)

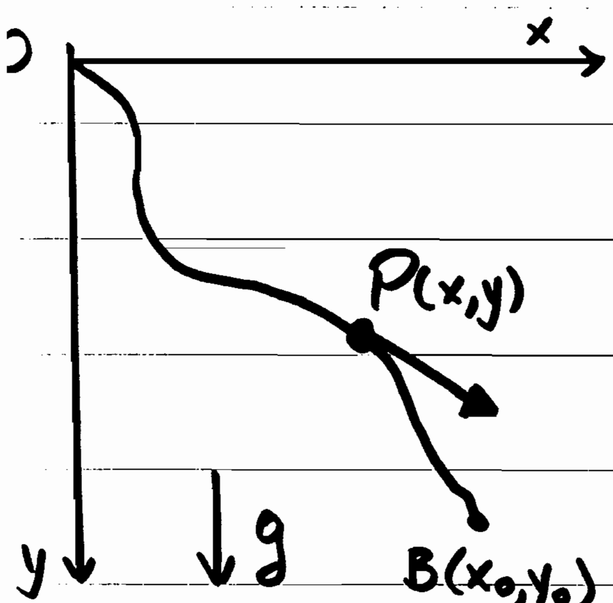
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = q \longleftarrow \partial_t \iiint_V \rho \, dV + \oiint_{\partial V} \rho u \cdot dS = \iiint_V q \, dV$$

P.D.E. form: needs differentiable functions

Integral form (discontinuous sols. also)

Differential equations also arise in the context of <u>extremizing</u> certain expressions (<u>functionals</u>).
Physical problems often formulated as <u>Variational Principles</u>

e.g. a bead slides down a wire w/o friction. Find shape of wire that gets bead from O to B fastest: ($y(x)$ unknown)



minimize $T = \dfrac{1}{\sqrt{2g}} \displaystyle\int_0^{x_0} \sqrt{\dfrac{1+(y')^2}{y}}\, dx$

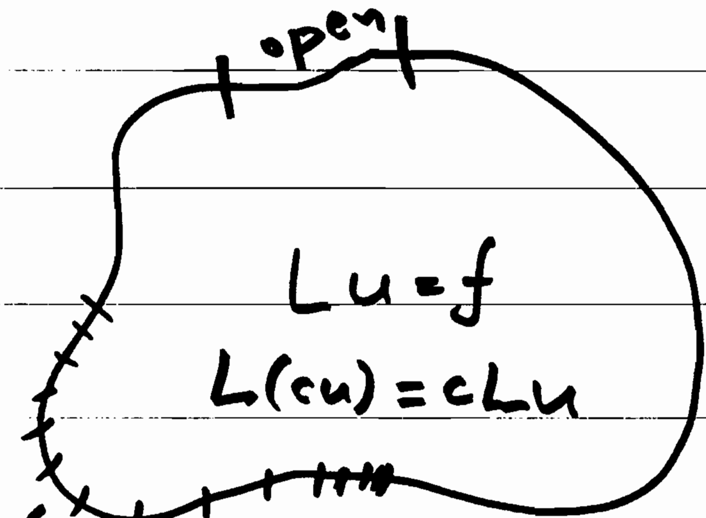
Calculus of Variations $\Rightarrow$ Euler-Lagrange D.E. (Ch. 17)

Boundary conditions

Complete specification of a problem in a finite or infinite domain requires relations on fixed boundaries, or conditions at ∞ (outgoing waves, boundedness)

radiation conditions

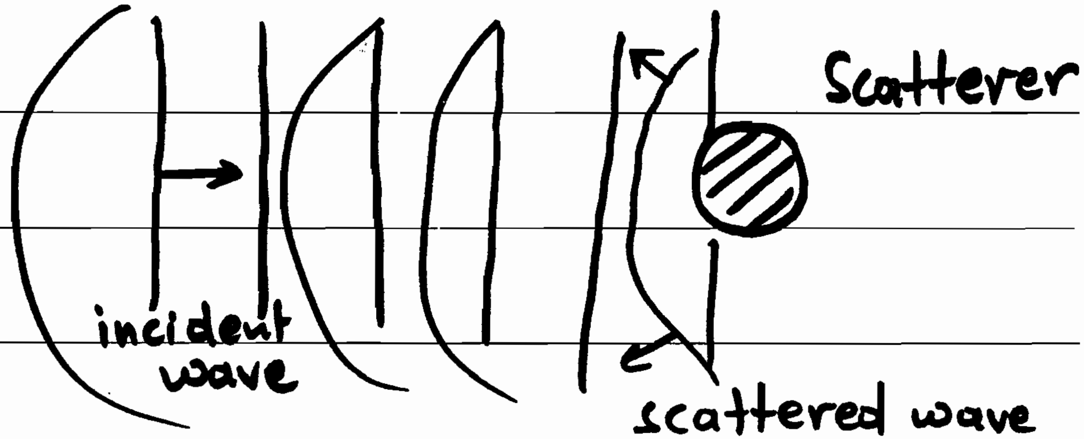
regularity conditions

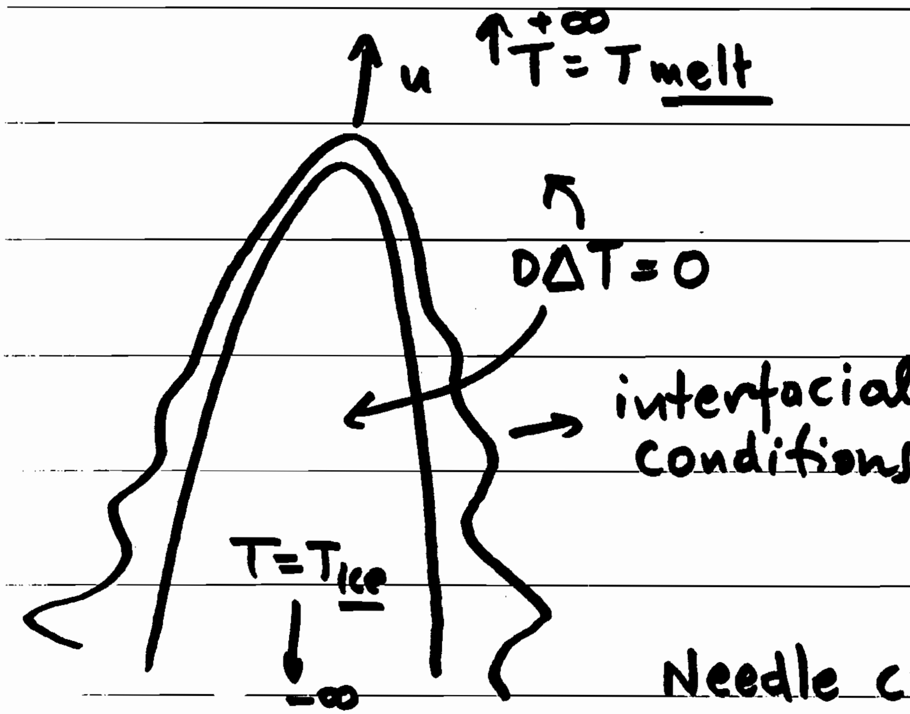


$\bar{n}\cdot\nabla u$ given (Neumann)

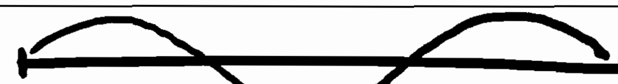
u given (Dirichlet)

* Fixed vs. moving (or open) boundaries

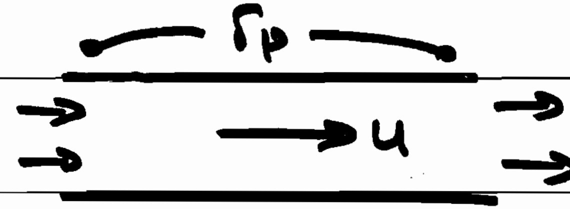




Needle cryst l



vibrating string



Channel

Methods of Solution

Exact: "closed" expressions which satisfy D.E. exactly.

Approximate: construct expressions that satisfy D.E. approximately, to within some tolerance (error).

Perturbation

Iteration

Numerical approximation

Approximate evaluation of exact solutions (asymptotics, acceleration of convergence)

Practical issues

Distinction between "exact" & "approximate" blurred when numerical expressions are sought,[(1)] but also when qualitative properties only are needed (2)

(1) Exact solutions often in form of infinite series or integrals

Frobenius (power) series : plug-and-solve (8.5)

4) Fourier series (trig. or generalized) (expansion in basis of function space)

Neumann series (successive approx. of integral equ. form)

11/1

* Questions of convergence (and rate of convergence, as only a few terms can be computed).

Fourier - & Laplace integrals (15)

$$\phi(x,t) = \int_{-\infty}^{\infty} F(k)\, e^{ik\cdot x - iW(k)t}\, dk$$

e.g. $W(k) = ck$: wave equ.

$W(k) = \gamma k^2$: elastic beam equ.

$W(k) = \sqrt{gk}$: deep water waves

Need to extract information re. propagation of disturbances: large time asymptotics

General Integral transforms

Bessel, Hilbert, Hankel ...

Green's functions: compute (8.7) solution due to "point-source", or "instantaneous disturbance", then find solution for general source or disturbance through superposition (leads to integral expressions - cf. variation of params.)

Contour Integral representations: D.E. are extended into the complex domain, and solutions are expressed using appropriate contours around "singularities" of DE. (useful for approximation).

Numerical Solution must often be used together with the various analytical techniques; must not be thought of as an alternative but rather as a variant.

Common problem: series converge slowly, need to approximate function to high accuracy: too many ops. lead to possible accumulation of roundoff errors.

$\Rightarrow$ need to accelerate convergence

$-1 \le x \le 1$

Ex. $\ln(1+x) = \sum_1^\infty (-1)^{n-1} \frac{x^n}{n}$

slow ($1/n$). But consider

$(1+\alpha_1 x)\ln(1+x) =$

$$= \sum_{n=1}^\infty (-1)^{n-1} \frac{x^n}{n} + \alpha_1 \sum_1^\infty (-1)^{n-1} \frac{x^{n+1}}{n}$$

$$= x + \sum_{n=2}^\infty (-1)^{n-1} \left[\frac{1}{n} - \frac{\alpha_1}{n-1}\right] x^n$$

$$\frac{n(1-\alpha_1)-1}{n(n-1)} \; ; \quad \alpha_1 = 1 : \frac{1}{n^2}$$

$\therefore$ new series has quadratic convergence (we return to this: Padé approximation).

Related issues: Chebyshev acceleration, Euler acceleratio

$$\ln(1+x) = \frac{\sum a_k x^k}{1+\alpha_1 x + \alpha_2 x^2 + \ldots}$$

14/1

Convergence questions arise also when considering Fourier or other Special function series:

e.g. when expanding discontinuous functions, coefficients decay slowly and many terms are needed for reasonable approximation:

$$g(x) = \sum_{-\infty}^{\infty} a_k e^{ikx} \longrightarrow g_K(x) = \sum_{k=-K}^{K} a_k e^{ikx}$$

$$\downarrow$$

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} g(x) e^{-ikx} dx$$

$a_k = \frac{1}{k}$

(Gibbs phenomenon)

<u>Fact</u>: if $g(x)$ is smooth, coefficients a_k decay (as $k \to \infty$) faster than any power of $(1/k)$. (<u>exponential decay</u>). Same is true for coefficients b_k in

$$\left.\begin{aligned} g(x) &= \sum_{0}^{\infty} b_k \phi_k(x) \\ b_k &= \frac{1}{N_k}\int_a^b g(x)\phi_k(x)\,w(x)\,dx \end{aligned}\right\}$$

$\phi_k(x)$: solutions of D.E. of form

$$\frac{d}{dx}\,p(x)\,\frac{d\phi_n}{dx} + \left(\lambda_n w(x) - q(x)\right)\phi_n(x) = 0$$

(<u>Sturm-Liouville problem</u>). (9)

Special functions

Trigonometric

* (11) Bessel, Hankel

(13.1) Hermite, Laguerre, Legendre, (12)

(13.3) Chebyshev, Jacobi

→ polynomials

Governing D.E. arise by separation of variables, (8.1) from basic PDE, in various curvilinear geometries

*: Unified as Hypergeometric functions (13.5, 4)

Complex variables

The location and type of zeroes of $p(x)$ in complex plane (i.e. $p(x) \to p(z)$, $z \in \mathbb{C}$) determine solution completely in cases of interest:

Regular vs. irregular (8.4) singular pts.

Frobenius theory allows the construction of power series (+ logarithmic or root singularities) to represent solutions of SL problems.

Integration around appropriate contours in $\mathbb{C}$ leads to contour integral representation of solutions.

Integral transforms can be applied directly to PDE (especially in infinite or semi-inf. domains) to reduce it (successively) to simpler forms.

Solution is again expressed in terms of complex (contour) integrals, even for (initially) real problems.

Complex variable techniques allow deducing information on solutions : asymptotics on integrals: Stationary phase, steepest descent, saddle point methods. (Sec. 7.4)

 20/1

Linear Operator theory
provides the theoretical
foundation for expansions in terms
of special functions – eigenfunctions
of Sturm-Liouville operators.
Extension to (∞-dimensional)
function space of ideas from linear algebra
and vector spaces:
(9.4) completeness, spectrum, basis,
eigen-function (-vector)
→ new concepts (∞-dim.) resolvent, continuous spectrum

21/1

Perturbation Theory: instead of approximating ~~exact~~ known solution, seek to approximate solution by solving "nearby" problems.

(Numerical solution: related)

1) Nonlinear problems: linearization

$$x'' + \omega^2 \sin x = f(t) \longrightarrow x'' + \omega^2 x = f(t)$$

"small oscillation"

frequency depends on amplitude
no superposition in general.

frequency indep. of amplitude.
Superposition