Complex Numbers

$$z = x+iy \sim (x,y) \sim \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

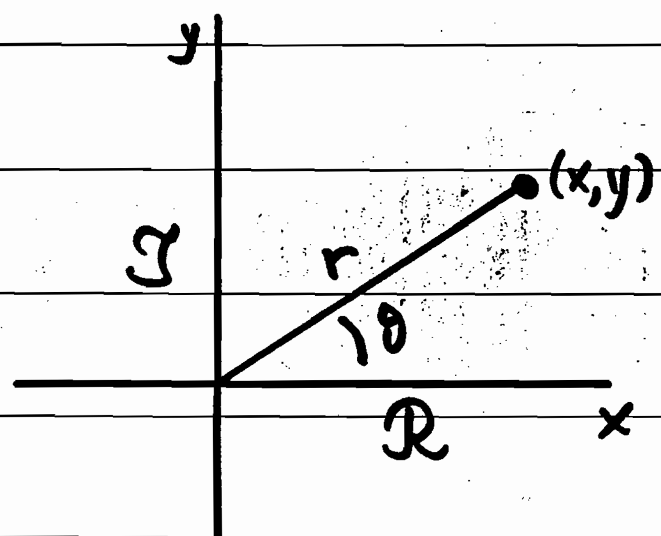
$$z_1+z_2 = (x_1+x_2)+i(y_1+y_2)$$

$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2+y_1x_2)$$

Polar form $z = r(\cos\vartheta + i\sin\vartheta) = re^{i\vartheta}$ Euler

$$z_1z_2 = r_1r_2(\cos\vartheta_1\cos\vartheta_2 - \sin\vartheta_1\sin\vartheta_2)$$
$$+ i r_1 r_2(\cos\vartheta_1\sin\vartheta_2+\sin\vartheta_1\cos\vartheta_2)$$
$$= r_1r_2\cos(\vartheta_1+\vartheta_2) + i r_1r_2 \sin(\vartheta_1+\vartheta_2)$$
$$= r_1e^{i\vartheta_1}\cdot r_2 e^{i\vartheta_2} = r_1r_2 e^{i(\vartheta_1+\vartheta_2)}$$

$$z^n = r^n(\cos\vartheta+i\sin\vartheta)^n = r^n e^{in\vartheta}$$
$$= r^n(\cos n\vartheta + i\sin n\vartheta) \quad \text{(de Moivre)}$$



ℂ - plane

Conjugate: $z^* = x - iy$

Functions $f(z) = u(x,y) + i\,v(x,y)$

* $e^{z} = e^{x}e^{iy} = e^{x}\cos y + i\,e^{x}\sin y$

* $\ln z = u + iv = w$

$\updownarrow$

$e^{w} = z \Rightarrow e^{u+iv} = x + iy = r\cos\vartheta + i r\sin\vartheta$

$\Rightarrow (e^{u}\cos v) + i\,(e^{u}\sin v) = r\cos\vartheta + i r\sin\vartheta$

$r = e^{u} \Rightarrow u = \ln r$ (real logarithm)

$\cos\vartheta = \cos v \Rightarrow v = \vartheta + 2n\pi$

$\therefore \ln z = \ln r + i\vartheta + i\,2n\pi, \quad n = 0, \pm1, \pm2, \ldots$

logarithm is a MULTIVALUED function

Principal value: choose argument in range $[0, 2\pi)$

i.e. $\operatorname{Log} z = \ln r + i\vartheta, \quad -\pi \le \vartheta < \pi$

Roots / Powers

$$z = re^{i\vartheta} = re^{i(\theta+2n\pi)};$$

$$e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi = 1, \forall n.$$

Polar representation entails ambiguity re. the argument (angle). This manifests itself when raising to powers:

$$z^\alpha = r^\alpha e^{i(\theta+2n\pi)\alpha}$$

(i) α integer : $e^{i2n\pi\alpha} = 1$

(ii) $\alpha = \frac{1}{m}$: $e^{i\frac{2n\pi}{m}}$; $n = 0, 1, \dots, m-1$ give distinct "roots";

but $e^{i\frac{2m\pi}{m}} = e^{i\cdot 0} = 1$

(iii) α irrational: get ∞-many distinct values as

$n = 0, \pm 1, \pm 2$

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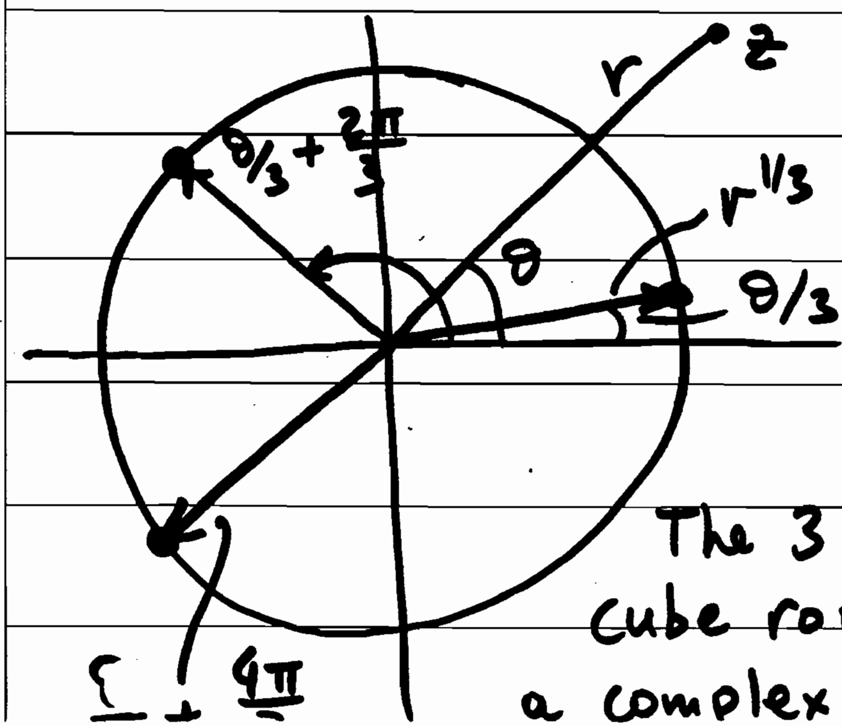
$$z^{1/3} = r^{1/3} e^{i(\theta + 2n\pi)/3}$$

$$= \begin{cases} r^{1/3} e^{i\theta/3} & n=0 \quad \rightarrow \text{Principal} \\ r^{1/3} e^{i\left(\theta/3 + \frac{2\pi}{3}\right)} & n=1 \\ r^{1/3} e^{i\left(\theta/3 + \frac{4\pi}{3}\right)} & n=2 \end{cases}$$

$$-\pi \le \theta < \pi$$

values repeat (e.g., $n = 7$, $n = 1$ give same value etc.).



The 3 distinct cube roots of a complex number

Ex. $(1+i)^{2-i}$

$$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\,e^{i\frac{\pi}{4}}$$

$$= e^{\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)}$$

$$(1+i)^{2-i} = e^{\frac{1}{2}\ln 2\,(2-i)}\; e^{i\left(2n+\frac{1}{4}\right)\pi(2-i)}$$

$$= e^{\ln 2 - \frac{i}{2}\ln 2}\; e^{i\left(4n+\frac{1}{2}\right)\pi}\; e^{\left(2n+\frac{1}{4}\right)\pi}$$

$$= 2e^{\left(2n+\frac{1}{4}\right)\pi}\; e^{i\left[\left(\cancel{4n}+\frac{1}{2}\right)\pi - \frac{1}{2}\ln 2\right]}$$

$$\therefore\ (1+i)^{2-i} = 2e^{\left(2n+\frac{1}{4}\right)\pi}\left\{\cos\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right) + i\sin\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right)\right\}$$

P.V. : $\underline{n=0}$

etc.

Trig. functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

<u>Ex</u>: $\cos z = 2 \Rightarrow \dfrac{w + \frac{1}{w}}{2} = 2$

$$\Rightarrow w^2 - 4w + 1 = 0 \quad ; \quad w = e^{iz}$$

$$\Rightarrow w = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3} = e^{iz}$$

$$\Rightarrow iz = \ln(2 \pm \sqrt{3}) + i\,2n\pi$$

$$z = -i\ln(2 \pm \sqrt{3}) + 2n\pi; \quad n = 0, \pm 1, \pm 2, \ldots$$

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Vector Analysis - key formulas (2d)

$\underline{F} = u(x,y)\,\hat{x} + v(x,y)\,\hat{y}$; vectorfield

$\phi(x,y)$ scalar field

$\nabla\phi = \partial_x \phi\,\hat{x} + \partial_y \phi\,\hat{y}$; gradient

$\nabla\cdot\underline{F} = \partial_x u + \partial_y v$ divergence

$\nabla\times\underline{F} = (\partial_x v - \partial_y u)\,\hat{z}$ curl

$\nabla\times(\nabla\phi) \equiv 0$; $\nabla\cdot(\nabla\times F) \equiv 0$; $\Delta\phi = \partial_x^2\phi + \partial_y^2\phi$ Laplacian

Line integral: $\int_c \underline{F}\cdot dr = \int_{t_1}^{t_2} \underline{F}(x(t),y(t))\cdot(\dot{x},\dot{y})\,dt$

$\int_{1\,c}^{2} \nabla\phi\cdot dr = \phi(2) - \phi(1)$; $\oint_c F\cdot dr = \iint \nabla\times F\cdot dS$ 2d

(independent of path)

$\oint u\,dx + v\,dy = \iint\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$

(Stokes)

Derivative of complex function

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}$$

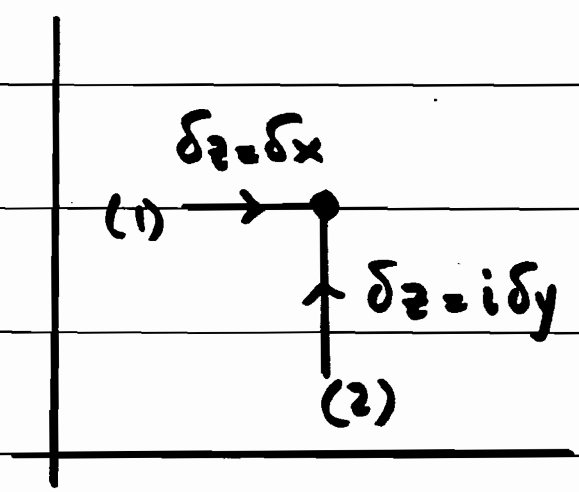
$$\delta f = \delta u + i \delta v$$

$$(1)\ \delta z = \delta x\ ; \quad f' = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right)$$

$$= \partial_x u + i \partial_x v$$

$$(2)\ \delta z = i \delta y\ ; \quad f' = \lim_{\delta y \to 0} \left(\frac{1}{i} \frac{\delta u}{\delta y} + \frac{i}{i} \frac{\delta v}{\delta y} \right)$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$



Derivative independent of limit process: equate real/imaginary parts:

$$\partial_x u = \partial_y v \quad (1) \quad ; \quad \partial_y u = - \partial_x v$$

Cauchy-Riemann equations

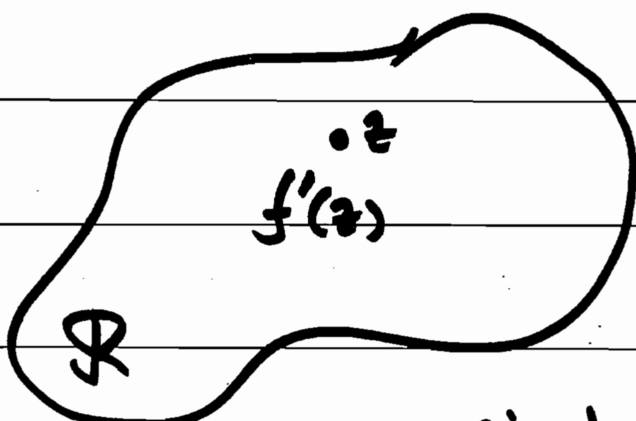
<u>$f(z)$ analytic</u> in region $\mathcal{R}$:

$f'(z)$ exists at every point of $\mathcal{R}$.

(in fact, we will prove that then $f(z)$ is ∞-times differentiable in $\mathcal{R}$).

Then

$$\begin{aligned}\Delta u &= u_{xx} + u_{yy} \\ &= (u_x)_x + (u_y)_y \\ &= (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0.\end{aligned}$$



$\therefore \Delta u = 0$: u harmonic

Similarly $\Delta v = 0$: v harmonic

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$f(z)$ entire : $f'(z)$ exists $\forall z \in \mathbb{C}$

singular point z_0 : $f'(z)$ does not exist at z_0

Ex. $f(z) = z^* = x - iy$ nowhere analytic

$$\partial_x(x) = 1 \neq \partial_y(-y) = -1$$

Ex. $u(x,y) = x^2 - y^2$ harmonic:

$(\partial_{xx} + \partial_{yy})(x^2 - y^2) = 2 - 2 = 0$; find v so $u + iv$ analytic.

Then

$$\left.\begin{aligned} u_x &= 2x = v_y \Rightarrow v = 2xy + g(x) ; \\ u_y &= -2y = -v_x \Rightarrow v = 2xy + h(y) \end{aligned}\right\} \Rightarrow g(x) = h(y) = \text{const.}$$

$$\Rightarrow v(x,y) = 2xy + c$$

Thus $f(z) = x^2 - y^2 + i2xy + ci$ $(= z^2 + ci)$ analytic

Meaning of derivative

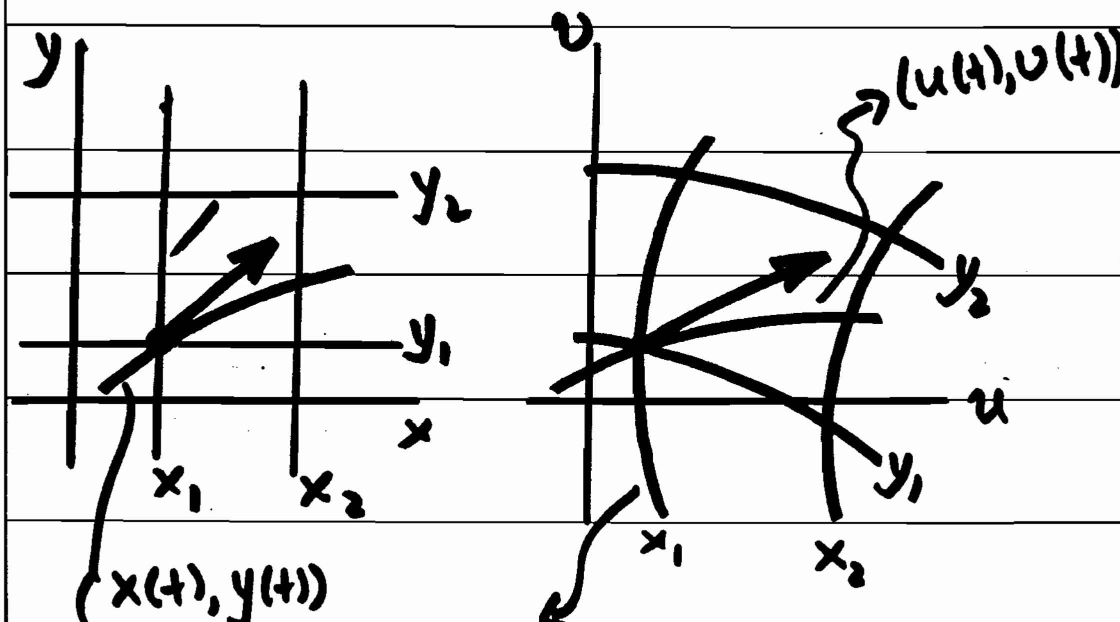
$$f(z) = u(x,y) + i\,v(x,y)$$

$$f'(z) = u_x + i v_x = u_x - i u_y = v_y + i v_x$$

$$|f'|^2 = \underset{f'}{(u_x - i u_y)}\,\underset{f'^*}{(v_y + i v_x)}$$

$$= u_x v_y - u_y v_x = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Jacobian
(i.e. area magnification)



$$\begin{pmatrix} \dot u \\ \dot v \end{pmatrix} = \begin{pmatrix} u_x \dot x + u_y \dot y \\ v_x \dot x + v_y \dot y \end{pmatrix}$$

$$\Downarrow$$

$$\dot u + i \dot v = (u_x + i v_x)(\dot x + i \dot y)$$

$$\rho e^{i(\theta+\Theta)} \quad = \quad R e^{i\Theta} \quad r e^{i\theta}$$

$u = u(x_1, y)$
$v = v(x_1, y)$ } etc.

angle of rotation