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RECAP

* $f(z) = u(x,y) + i\,v(x,y)$ <u>analytic</u>

  (1) $f'(z)$ exists in region $\mathcal{R}$

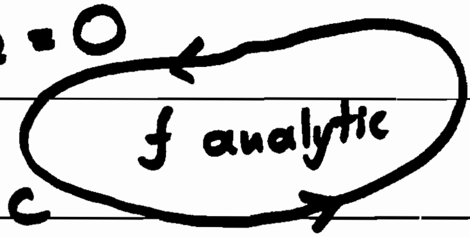
  (2) C.R. equs: $u_x = v_y$, $u_y = -v_x$ in $\mathcal{R}$

* <u>singular points</u>: places of non-analyticity

* CAUCHY INTEGRAL THEOREM: $\oint_c f(z)\,dz = 0$

  C closed path; f analytic in C.

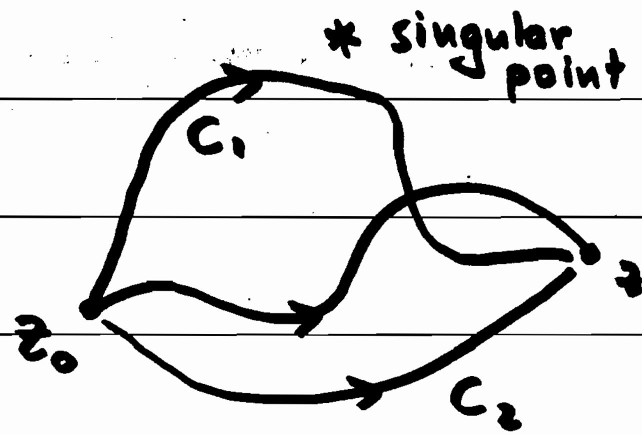
$\oint_c f\,dz = 0$



* If C_1 can be deformed into C_2 without crossing any singular points of f

$$\int_{z_0, C_1}^{z} f(z)\,dz = \int_{z_0, C_2}^{z} f(z)\,dz$$

(equivalent paths)

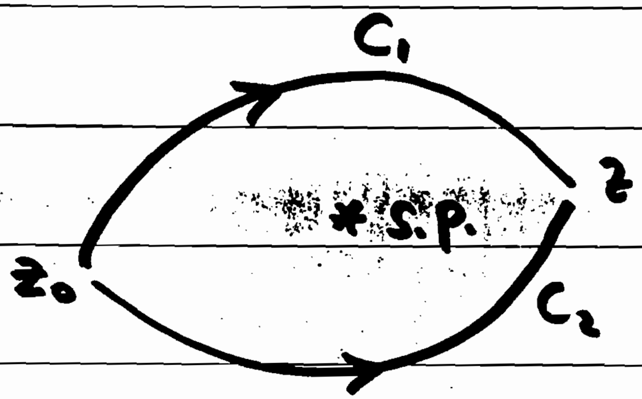


*** Antiderivative** if $f = F'$ in $\mathcal{R}$

then $\oint_C f\,dz = 0$ for any path in $\mathcal{R}$

and $\int_{z_0}^{z} f(z)\,dz = F(z) - F(z_0)$

for any path joining z_0 to z (and not containing s.p.).



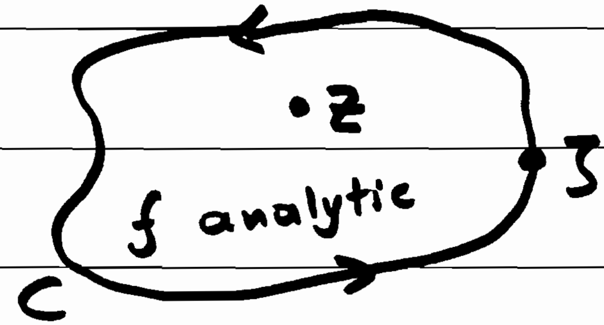
(possibly) inequivalent paths

Note: if $F(z)$ is multivalued (like $\ln z$), then different paths may lead to different values of the integral.

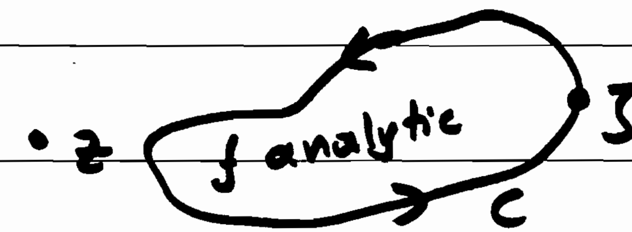
(more when we discuss branch points).

CAUCHY INTEGRAL FORMULA

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)\,d\zeta}{\zeta - z} \quad (\mathrm{I})$$

(I) 

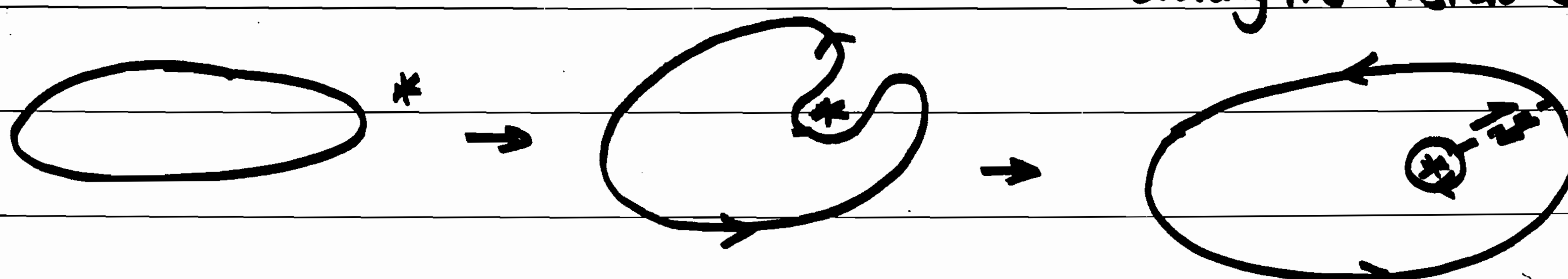
z in C

II 

z outside

$$\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = 0; \quad \frac{f(\zeta)}{\zeta - z} \text{ is}$$

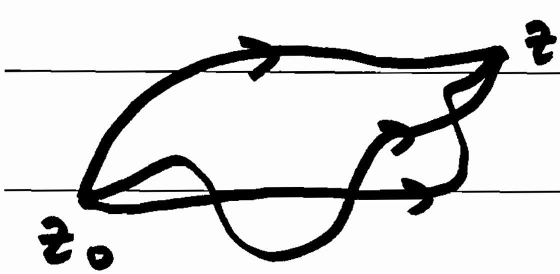
analytic inside C.



Rules for deforming contours

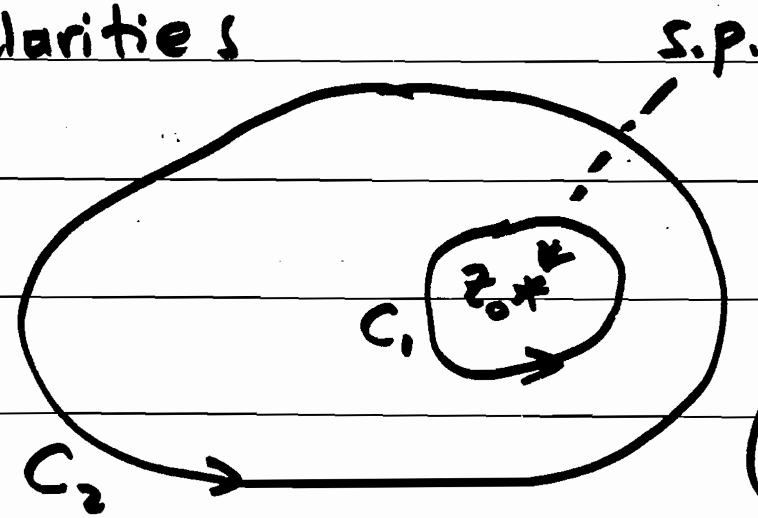
(1) Open ended: preserve end-points

move contour w/o crossing singularities



$\int_{z_0}^{z} f(z)\,dz$ same for all these.

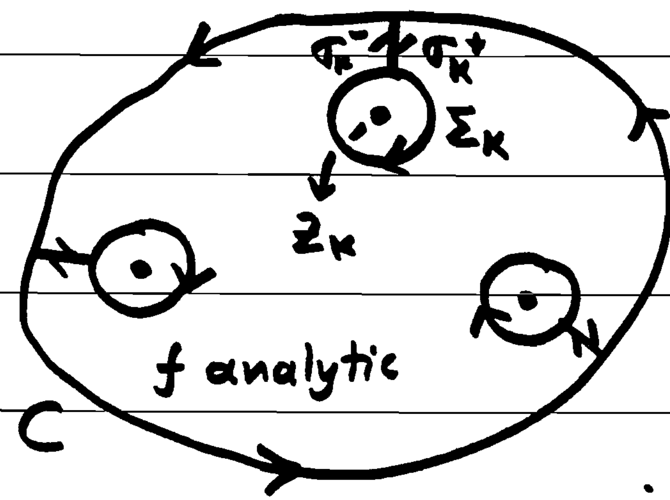
(2) Closed contours: move contour around without crossing singularities



$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

($f(z)$ analytic in region between C_1, C_2)

Several singularities



Each Σ_k contains only one (isolated) s.p., z_k.

Can shrink the paths Σ_k to infinitesimal circles about z_k.

f is analytic in region enclosed by composite contour

$$C + \sum_{k=1}^{n} \Sigma_k + \sum_{k=1}^{n} (\sigma_k^+ + \sigma_k^-) \quad ; \text{ so}$$

cancel

$$\left\{ \oint_C - \sum_{k=1}^{n} \oint_{\Sigma_k} + \sum_{k=1}^{n} \cancel{\left(\oint_{\sigma_k^+} + \oint_{\sigma_k^-} \right)} \right\} f(z)\,dz = 0$$

$$\Rightarrow \oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{\Sigma_k} f(z)\,dz$$

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Taylor expansion

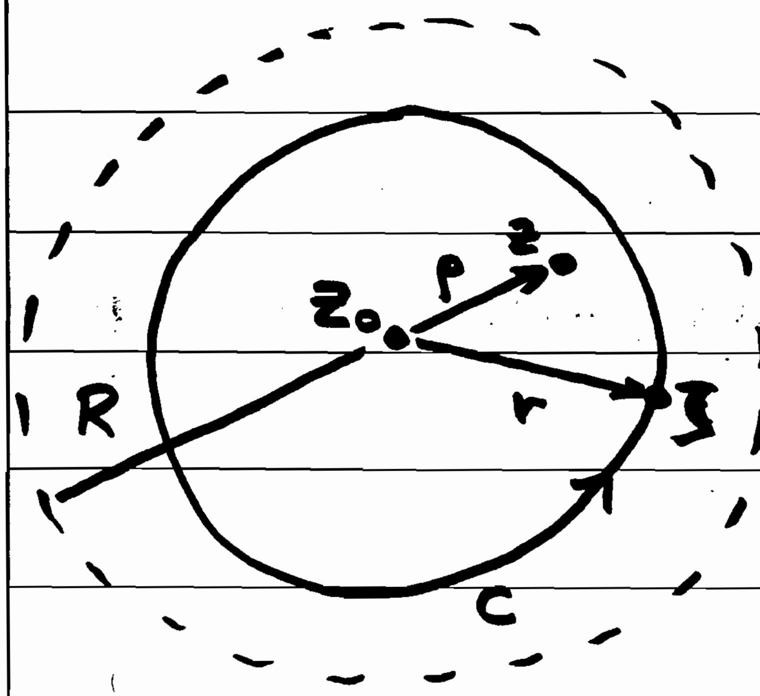
$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

express in terms of values at $z = z_0$.

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - \zeta_0)} = \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]}$$

$$= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \cdots + \left(\frac{z - z_0}{\zeta - z_0}\right)^n + \cdots\right]$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$



ζ, $|\zeta - z_0| = r$

$|z - z_0| = \rho$

$\rho < r < R$

will see:

$$\frac{1}{1 - \zeta} = 1 + \zeta + \zeta^2 + \cdots + \zeta^n + \cdots$$

uniformly convergent for $|\zeta| \le \delta < 1$

$$\frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{n=0}^{\infty}{}' (z-z_0)^n \left[\frac{1}{2\pi i}\oint \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta\right]$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}{}' \frac{f^n(z_0)}{n!}(z-z_0)^n$$

Converges so long as Cauchy integral formula is valid, i.e. if contour C does not contain singularities of $f(z)$. (Can show that radius of convergence is exactly distance from z_0 to nearest singularity of $f(z)$.)

Schwarz reflection principle

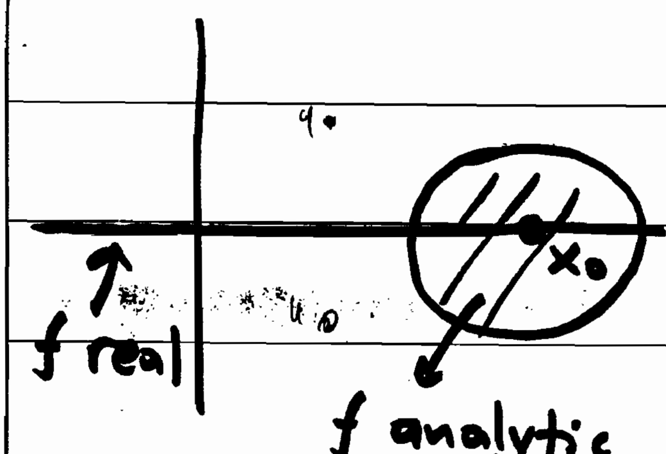
let $g(z) = (z - x_0)^n$; then

$$g^*(z) = (z^* - x_0)^n = g(z^*).$$

In general, if $f(z)$ is real for $z = x$ (on real axis) and analytic in a region that contains the real axis, then

$$f^*(z) = f(z^*) \quad ; \; z^* = x - iy$$

$$\left(\text{i.e. } f^*(z) = u(x,y) - iv(x,y) = u(x,-y) + iv(x,-y) = f(z^*) \right)$$



$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (z - x_0)^n ; \quad f^n(x_0) \text{ real}$$

Then $f^*(z) = \sum \frac{f^{(n)}(x_0)}{n!} (z^* - x_0)^n = f(z^*)$ in disk. Can show that property extends to entire region of analyticity

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Digression: Infinite series

$S = \sum_0^\infty b_k$; let $b_k \geq 0$

Partial sum $S_N = \sum_0^N b_k \xrightarrow{N\to\infty} S < \infty$
convergence

else, if $S_N \xrightarrow{N\to\infty} \infty$, divergence.

COMPARISON TEST:

$\sum a_k$ converges / diverges, $b_k \lesseqgtr a_k \Rightarrow \sum b_k$ converges / diverges

Ex: $S = \sum r^k$

$S_N = 1 + r + \dots + r^N = \frac{1-r^{N+1}}{1-r} \xrightarrow{N\to\infty} \begin{cases} \infty, & r \geq 1 \\ \frac{1}{1-r}, & r < 1 \end{cases}$

geometric series

Ex. $\sum_1^\infty \frac{1}{k} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq\, 2\cdot\frac{1}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq\, 4\cdot\frac{1}{8}} + \underbrace{\left(\frac{1}{9} + \dots + \frac{1}{16}\right)}_{\geq\, 8\cdot\frac{1}{16}} + \dots \to \infty$

diverges.

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Convergence:

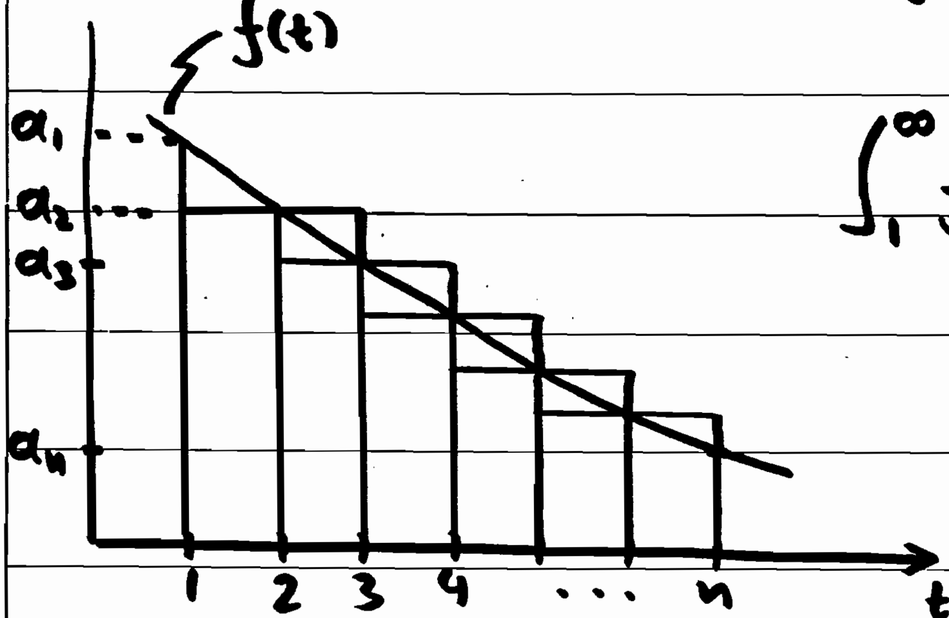
Root test: $(a_n)^{1/n} \leq r < 1$ eventually

(i.e. $\forall n > N$ for some N)

(then $a_n \leq r^n$; comparison with geo)

Ratio test: $\frac{a_{n+1}}{a_n} \leq r < 1$ eventually

Cauchy Integral test : $f(n) = a_n$



$$\int_1^\infty f(x)\,dx \leq \sum_1^\infty a_n \leq a_1 + \int_1^\infty f(x)\,dx$$

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General series (also complex terms)

<u>Absolute convergence</u>:

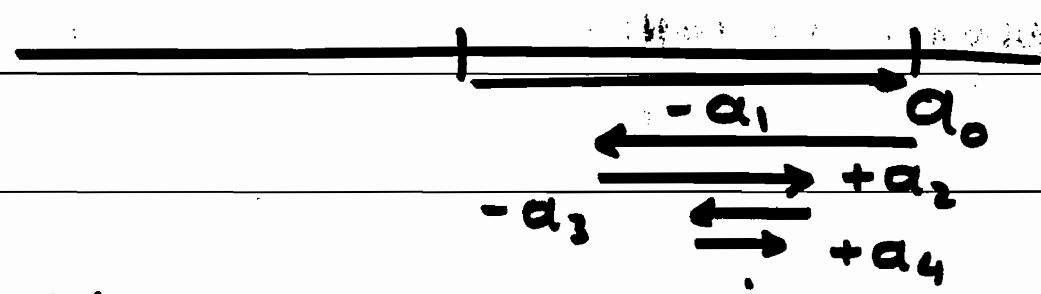
$\sum |b_n|$ converges $\Rightarrow$ $\sum b_n$ converges

(converges absolutely)

<u>Alternating series</u>: $a_n > 0$, $a_n \searrow 0$ <u>eventually</u>

$\sum_{0}^{\infty} (-1)^n a_n$ converges

(but not necessarily absolutely)



($\sum^{\infty} (-1)^{n+1} \frac{1}{n}$ converges - but not absolutely!)

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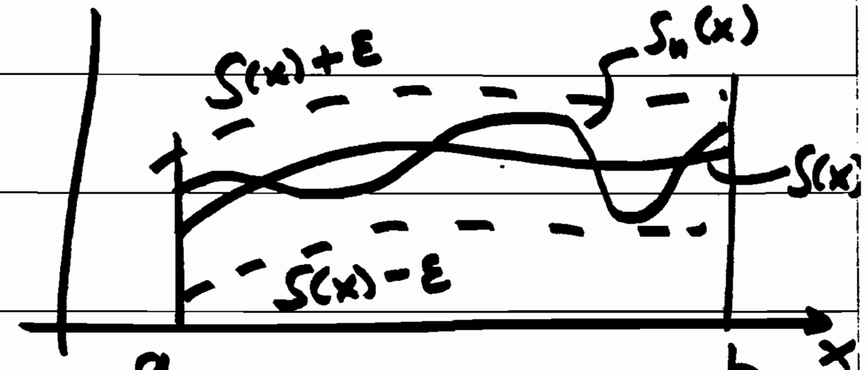
Series of variable terms

$$S(x) = \lim_{n\to\infty} s_n(x) \; ; \; s_n(x) = \sum_{0}^{n} f_k(x)$$

Uniform convergence

$x \in [a,b]$



ε-tube about $S(x)$

$\forall \varepsilon > 0$, $\exists N$, indep. of x

such that, for $n > N$, $|S(x) - s_n(x)| < \varepsilon$,

i.e. tail $\left| \sum_{n+1}^{\infty} f_k(x) \right| < \varepsilon$

Weierstrass M-test : $M = \sum_{1}^{\infty} M_k$, $|f_k(x)| \le M_k$, $x \in [a,b]$

Then, if $M < \infty$, $\sum f_k(x)$ is uniformly a[illegible] absolutely convergen[illegible]

Notes: (i) $\sum f_n(x)$ uniformly convergent

then $\int \sum f_n = \sum \int f_n$

↑

(ii) $f_n(x)$ continuous; $\sum f_n$ unif. conv

$\Rightarrow S(x)$ continuous

(iii) $\sum f_n'$ uniformly convergent, then

$\sum f_n' = S'$. (f_n' continuous).

Power series: $\sum_0^\infty a_n z^n$; If $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = R$ exits, then

R is "radius of convergence"

i.e. series converges for $|z| < R$ (uniform for $|z| \le r < R$)

Inside disk of convergence, series can be differentiated o[r] integrated term-by-[t]erm

Example: $I(\alpha) = \oint_{|z|=1} \frac{e^{\alpha z}}{z^3} dz$

(i) Cauchy int. form.: $I(\alpha) = \frac{2\pi i}{2!} \cdot \frac{d^2}{dz^2} \left(e^{\alpha z}\right)\Big|_{z=0}$

$= \pi i \alpha^2$

(ii) $e^{\alpha z} = 1 + \alpha z + \frac{(\alpha z)^2}{2} + \dots = \sum_0^\infty \frac{(\alpha z)^n}{n!}$

$$I(\alpha) = \sum_0^\infty \frac{\alpha^k}{k!} \underbrace{\oint_{|z|=1} z^{k-3} dz}_{= 2\pi i \cdot \delta_{k,2}} = 2\pi i \cdot \frac{\alpha^2}{2} = \pi i \alpha^2$$